Applied Mathematics and Computation 320 (2018) 452–463

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Hybrid methods for direct integration of special third order ordinary differential equations

Y. D. Jikantoro [a,c,*], F. Ismail [a,b], N. Senu [a,b], Z.B. Ibrahim [a,b]

[a] Department of Mathematics, Universiti Putra Malaysia, Serdang 43400, Selangor, Malaysia
[b] Institute for Mathematical Research, Universiti Putra Malaysia, Serdang 43400, Selangor, Malaysia
[c] Department of Mathematics, Ibrahim Badamasi Babangida University, P.M.B 11, Lapai, Nigeria

ARTICLE INFO

Keywords:
Hybrid method
Three-step method
B-series
Order conditions
Third order ordinary differential equations
Numerical integrator

ABSTRACT

In this paper we present a new class of direct numerical integrators of hybrid type for special third order ordinary differential equations (ODEs), $y''' = f(x, y)$; namely, hybrid methods for solving third order ODEs directly (HMTD). Using the theory of B-series, order of convergence of the HMTD methods is investigated. The main result of the paper is a theorem that generates algebraic order conditions of the methods that are analogous to those of two-step hybrid method. A three-stage explicit HMTD is constructed. Results from numerical experiment suggest the superiority of the new method over several existing methods considered in the paper.

1. Introduction

Third order ordinary differential equation (ODE) is used in modeling problems arising in various areas of applied science such as biology, quantum mechanics, celestial mechanics and chemical engineering [1,2]. For instance, the models that describe acoustic wave propagation in relaxing media, draining coating flows e.t.c are all third order differential equations [1,3–8]. Some of these equations occur in a special form $y''' = f(x, y(x))$, for instance, thin film flow problem studied in [9] and the references therein.

There is always a shortfall in the part of analytical solutions that satisfy most of the ODEs, third order ODEs inclusive. Hence, the search for approximate solutions by numerical means becomes imperative. Over the last few decades, a lot of work has been done on the solutions of third order ODEs, especially in the area of linear multistep related methods. For instance, the P-stable linear multistep method by Awoyemi [10] and the hybrid collocation method by Awoyemi and Idowu [11]. More of these can be found in the works of Majid et al. [12], Olabode and Yusuph [13], Mahrkanoon [14], Guo et al. [5], Myers [6], Abdulmajid et al. [15] , Ken et al. [16] etc. and the references therein.

Traditionally, third order ODEs can be solved by first transforming them into systems of first order equations and applying Runge–Kutta (RK) methods or linear multistep methods, but this could be computationally costlier than the direct methods. In this paper, our main concern is with the initial value problems (IVPs) of special third order ODEs

$$y'''(x) = f(x, y(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0, \tag{1}$$

where $y \in R^d$, f: $R \times R^d \to R^d$ is a vector value function. The specialty associated with (1) is the fact that f does not depend on y', y'' explicitly. Inspired by Nyström methods (RKN), You and Chen [1] proposed a Runge–Kutta method (RKT) for solving

* Corresponding author at: Department of Mathematics, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia.
E-mail addresses: ydjikantoro@ibbu.edu.ng, jdauday@yahoo.ca (Y. D. Jikantoro).

https://doi.org/10.1016/j.amc.2017.10.003
0096-3003/

Table 1
General coefficients of HMTD.

-2	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$\cdots$	$a_{1,m}$
-1	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$\cdots$	$a_{2,m}$
0	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	$\cdots$	$a_{3,m}$
c_4	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	$\cdots$	$a_{4,m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_m	$a_{m,1}$	$a_{m,2}$	$a_{m,3}$	$\cdots$	$a_{m,m}$
	b_1	b_2	b_3	$\cdots$	b_m

(1) directly. Motivated by two-step hybrid method for solving special second order ODEs proposed by Coleman [17], we propose and investigate three-step hybrid method for solving (1) directly namely, HMTD.

Suppose we want to extend (1.3) of [17] in a way to solve (1) directly, see [18], we get

$$y_{n+1} = 2y_n - y_{n-1} + h^2\sum_{i=1}^{s}\bar{b}_i g(x, y(x)) + h^3\sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i),$$

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} + h^2\sum_{j=1}^{s}\bar{a}_{i,j} g(x, y(x)) + h^3\sum_{j=1}^{s} a_{i,j} f(x_n + c_j h, Y_j),$$

where $g(y(x)) = y''$ and $f(y(x)) = y'''$. Using Eqs. (4.1) and (5.1) of [17] and applying difference formula, we get

$$\begin{aligned}\mathbf{Y} &= \frac{1}{2}\Big\{\big(\mathbf{c}^2 + 3\mathbf{c} + 2\mathbf{e}\big)\otimes y_n - 2\big(\mathbf{c}^2 + 2\mathbf{c}\big)\otimes y_{n-1} + \big(\mathbf{c}^2 + \mathbf{c}\big)\otimes y_{n-2}\Big\} + h^3(\mathbf{A}\otimes\mathbf{I})f(x_n + \mathbf{c}h, \mathbf{Y}),\\ y_{n+1} &= 3(y_n - y_{n-1}) + y_{n-2} + h^3\big(\mathbf{b}^T\otimes\mathbf{I}\big)f(x_n + \mathbf{c}h\mathbf{Y}),\end{aligned} \tag{2}$$

in a vector notation for the case $y''' = f(x, y(x))$, which is the proposed HMTD method. Where $\mathbf{b} = [b_1, \ldots, b_m]^T$, $\mathbf{c} = [c_1, \ldots, c_m]^T$, $\mathbf{e} = [1, \ldots, 1]^T$, $\mathbf{A} = [a_{i,j}]^T$, $\mathbf{Y} = [Y_1, \ldots, Y_m]^T$ and $\mathbf{I}$ is identity matrix of $m\times m$ dimension. The coefficients of the methods are summarized in Table 1.

In Section 2, we present the theory of B-series and the associated rooted trees through which order conditions of the proposed method are derived. Local truncation error and order of convergence of the method are presented in Section 3. We present algebraic order conditions of the method in Section 4. As an example, explicit 3-stage HMTD is presented in Section 5. Numerical experiment is presented in Section 6. And conclusion is presented in Section 7 of the paper.

2. B3-series and associated rooted trees

It is customary to consider the autonomous case of (1) when working on the order conditions of HMTD methods, as in the case of RK methods, RKN methods and RKT methods for first, second and third order ODEs methods, respectively.

$$y'''(x) = f(y(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0. \tag{3}$$

Continuous differentiation of the exact solution $y(x)$ with respect to x gives the following:

$$\begin{aligned} y' &= y',\ y'' = y'',\ y''' = f(y),\ y^{iv} = f'(y)y',\ y^{v} = f''(y)(y', y')\\ &\quad + f'(y)y'',\ y^{vi} = f'''(y)(y', y', y') + 3f''(y)y'y'' + f'(y)f(y),\\ y^{vii} &= f^{iv}(y)(y', y', y', y') + 6f'''(y)(y', y', y'') + 3f''(y)(y'', y'')\\ &\quad + 4f''(y)(f(y), y') + f'(y)f'(y)y'.\end{aligned}$$

We shall use the tri-coloured trees introduced in [1] throughout in this paper. The relevant tri-coloured trees consist of vertices: “meagre”, “fat black” and “fat white” vertices representing y', y'', and f, respectively, which are connected by the tree ‘branches’. A meagre vertex is represented by a small dot (.), fat black vertex by a big dot (•) and fat white vertex by a small circle (○). When a line, which in this case represents a branch of tree, is leaving a vertex, it indicates a partial differentiation. The destination vertex of the line determines with respect to which component is the differentiation. The differentiation is with respect to the component of y if the destination vertex is meagre, it's with respect to the component of y' if the destination vertex is fat black and it's with respect to the component of y'' if the destination vertex is fat white. This implies that fat white vertex has only meagre sons since f does not depend on y' and y''. On the other hand, fat black vertex has only one son and this son must be fat white, because y'' has only one non-zero derivative with respect to itself and has non with respect to y' neither with y. Furthermore, meagre vertex has only one son and the son must be fat black due to the fact that it has no derivative with respect to y nor y''. All these constitute the rules for generating the rooted trees associated with special third order ordinary differential equations. The summary of the rules can be found in [1]. The following definitions are adopted from [1,17].