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Exponential stability of linear delayed differential systems

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ABSTRACT

Linear delayed differential systems

$$\dot{x}_i(t) = -\sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(t)x_j(h_{ij}^k(t)),\ \ i = 1, \dots, m$$

are analyzed on a half-infinity interval $t \geq 0$. It is assumed that m and r_{ij}, $i, j = 1, \dots, m$ are natural numbers and the coefficients $a_{ij}^k : [0, \infty) \to \mathbb{R}$ and delays $h_{ij}^k : [0, \infty) \to \mathbb{R}$ are measurable functions. New explicit results on uniform exponential stability are derived including, as partial cases, recently published results.

1. Introduction

There are two main forms of linear systems of delay differential equations. The first one is a scalar form

$$\dot{x}_i(t) = -\sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(t)x_j(h_{ij}^k(t)),\ \ i = 1, \dots, m, \tag{1}$$

where $t \geq t_0 \geq 0$, $t_0 \in \mathbb{R}$, m and r_{ij}, $i, j = 1, \dots, m$ are natural numbers, coefficients $a_{ij}^k : [0, \infty) \to \mathbb{R}$ and delays $h_{ij}^k : [0, \infty) \to \mathbb{R}$ are given functions, and $x_i : [-h, \infty) \to \mathbb{R}$, where

$$h := \inf_{t\geq 0} \min_{\substack{i,j=1,\dots,m\\ k=1,\dots,r_{ij}}} \{h_{ij}^k(t)\},$$

are unknown functions.

The second one is a vector form

$$\dot{X}(t) = -\sum_{k=1}^{m} A_k(t)X(h_k(t)), \tag{2}$$

where $t \geq t_0 \geq 0$, $t_0 \in \mathbb{R}$, m is a natural number, $A_k = (a_{ij}^k)_{i,j=1}^m : [0, \infty) \to \mathbb{R}^{2m}$ are matrix functions, $h_k : [0, \infty) \to \mathbb{R}$ are delay functions, and

$$X = (x_1, \dots, x_m)^T : [h^*, \infty) \to \mathbb{R}^m,$$

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where

$$h^* := \inf_{t\geq 0}\min_{i=1,\dots,m}\{h_i(t)\},$$

is an unknown vector function.

Papers on stability of linear differential delayed system can also be divided into two groups. The larger one considers vector form (2) (see for example papers [1–6] and monographs [7,8]) while the smaller one is devoted to scalar systems (1), we refer to papers [9–13] and to a monograph [14].

In the paper, we will study scalar systems (1). First, let us review some known stability results. In order to simplify this review, we consider non-autonomous systems and do not include the results specified for autonomous systems only (for such systems, we refer to interesting papers [11,13]). We also do not consider results obtained under the assumption

$$h_{ii}^k(t) \equiv t, \quad i = 1, \dots, m, \quad k = 1, \dots, r_{ii}.$$

In the sequel, we frequently use the concept of a non-singular M-matrix. For the sake of convenience, we recall this notion.

Definition 1 ([15]). An $m \times m$ matrix $G = (g_{ij})_{i,j=1}^m$ is called a non-singular M-matrix if $g_{ij} \leq 0$, $i, j = 1, \dots, m$, $i \neq j$ and one of the following equivalent conditions holds:

1. There exists a positive inverse matrix G^{-1}.
2. All the principal minors of matrix G are positive.

In [12], the authors consider a non-autonomous system, which is a partial case of system (1),

$$\dot{x}_i(t) = -\sum_{j=1}^{m} a_{ij}(t)x_j(h_{ij}(t)), \quad i = 1, \dots, m, \tag{3}$$

where $t \in [t_0, \infty)$, $a_{ij}(t)$, $h_{ij}(t)$ are continuous functions, $h_{ij}(t) \leq t$, and $h_{ij}(t)$ are monotone increasing functions such that $\lim_{t\to\infty} h_{ij}(t) = \infty$, $i, j = 1, \dots, m$.

Theorem 1. *[12, Theorem 2.2] Assume that, for $t \geq t_0$, there exist non-negative numbers b_{ij}, $i, j = 1, \dots, m$, $i \neq j$ such that $|a_{ij}(t)| \leq b_{ij}a_{ii}(t)$, $i, j = 1, \dots, m$, $i \neq j$, $a_{ii}(t) \geq 0$ and*

$$\int^{\infty} a_{ii}(s)ds = \infty, \qquad d_i = \limsup_{t\to\infty} \int_{h_{ii}(t)}^{t} a_{ii}(s)ds < 3/2, \qquad i = 1, \dots m.$$

Let, for the entries $\tilde{b}_{ij}$, $i, j = 1, \dots, m$ of an $m \times m$ matrix $\tilde{B}$,

$$\tilde{b}_{ii} = 1, \quad i = 1, \dots, m$$

and, for $i \neq j$, $i, j = 1, \dots, m$,

$$\tilde{b}_{ij} = \begin{cases} -\left(\dfrac{2+d_i^2}{2-d_i^2}\right)b_{ij}, & \text{if} \quad d_i < 1, \\[2ex] -\left(\dfrac{1+2d_i}{3-2d_i}\right)b_{ij}, & \text{if} \quad d_i \geq 1. \end{cases}$$

If $\tilde{B}$ is a non-singular M-matrix, then system (3) is asymptotically stable.

Theorem 1 is an object of attention since, for a scalar case it coincides with the well-known 3/2 stability condition (we refer, e.g., to [16–18]). Nevertheless, we point out the following restrictions in Theorem 1. The result is valid only for systems with continuous coefficients and delays and the proof of the theorem depends on this assumption. Moreover, any equation of system (3) includes only one diagonal term. It means that system (3) is a partial case of (1) for $r_{ij} = 1$, $i, j = 1, \dots, m$. Besides, delays in system (1) must be monotone increasing functions.

In monograph [14], the following result for the system

$$\dot{x}_i(t) = -\sum_{k=1}^{m}\sum_{j=1}^{n} a_{ij}^k(t)x_j(h_k(t)), \quad i = 1, \dots, n \tag{4}$$

is proved. In its formulation, we use the notation $a^+ = \max\{a, 0\}$, $a \in \mathbb{R}$ and, for a function f essentially bounded on $[t_0, \infty)$, we use the norm

$$\|f\|_{L_\infty} = \operatorname{ess\,sup}_{t\geq t_0}\|f(t)\|.$$