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A class of accelerated parameterized inexact Uzawa algorithms for complex symmetric linear systems☆

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ABSTRACT

We establish a class of accelerated parameterized inexact Uzawa (APIU) algorithms for solving the complex symmetric linear systems. Our main contribution is accelerating the convergence of the PIU algorithm by making use of the extrapolation technique which is based on the eigenvalues of the iterative matrix. These accelerated parameterized inexact Uzawa algorithms involve two iteration parameters whose special choices can recover the parameterized inexact Uzawa algorithm and some other methods. First, the accelerated model for the PIU algorithm is established and the accelerated PIU algorithm is presented. Then we study the convergence of the corrected PIU algorithm. Moreover, we present the optimal iteration parameter and the corresponding optimal convergence factor for the PIU method. We also consider acceleration of the PIU iteraton by Krylov subspace methods. Numerical experiments are presented to illustrate the theoretical results and examine the numerical effectiveness of the new method.

1. Introduction

Let n be a positive integer. We consider the iterative solution of systems of linear equations of the form

$$Ax = b, \quad A \in C^{n\times n} \quad \text{and} \quad x, b \in C^n, \tag{1.1}$$

where $A \in C^{n\times n}$ is a complex symmetric matrix of the form

$$A = W + \mathrm{i}T, \tag{1.2}$$

and $W, T \in R^{n\times n}$ are real, symmetric, and positive definite matrices. Here and in the sequel, we use $\mathrm{i} = \sqrt{-1}$ to denote the imaginary unit. Owing to the positive definite nature of the matrices T and W, we know that A is non-Hermitian.

Complex symmetric linear systems of this kind arise in many important problems in scientific computing and engineering applications, including diffuse optical tomography [1], FFT-based solution of certain time-dependent PDEs [2], quantum mechanics [3], algebraic eigenvalue problems [4,5], structural dynamics [6], and lattice quantum chromodynamics [7]. For more details about the practical backgrounds of this class of problems, we refer to [8–13] and the references therein.

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For solving the complex symmetric linear system (1.1) efficiently, some effective matrix-splitting-type iterative methods have been proposed, see [8–14,26,27,40] et al. Recently, a parameterized splitting iteration method for complex symmetric linear systems was proposed in [15] by Zhang and Zheng. In [16], Bai et al. studied an efficient iterative method for the large sparse non-Hermitian positive definite system of linear equations based on the Hermitian and skew-Hermitian splitting of the coefficient matrix. Due to its promising performance and elegant mathematical properties, the HSS scheme immediately attracted considerable attention, resulting in numerous papers devoted to various aspects of the new algorithm. So we can apply the HSS iteration method [16] or its preconditioned variant PHSS (i.e., the preconditioned HSS, see [17]) which were proposed by Bai et al. to compute an approximate solution of the linear system (1.1). In addition, the convergence properties of the PHSS method can be found in [18]. In [19,20], Bai et al. further generalized the technique for constructing HSS iteration method for solving large sparse non-Hermitian positive definite system of linear equations to the normal and skew-Hermitian (NS) splitting obtaining a class of normal and skew-Hermitian splitting (NSS) iteration methods. Theoretical analyses shown that the NSS iteration method converges unconditionally to the exact solution of the system of linear Eq. (1.1). A potential difficulty with the HSS iteration approach is the need to solve the shifted skew-Hermitian sub-system of linear equations at each iteration step. Hence, Bai et al. presented a modification of the HSS (MHSS) iteration scheme in [12] and some of its basic properties are studied. Moreover, in [21], the authors proposed a preconditioned variant of the modified HSS (PMHSS) iteration method for solving the complex symmetric systems of linear equations, and they also discussed the spectral properties of the PMHSS-preconditioned matrix in the paper.

In this paper, we present a class of accelerated parameterized inexact Uzawa iteration methods for solving the complex symmetric linear system (1.1). And we call this new algorithm APIU method for simplicity. The APIU method is based on the extrapolation method and the eigenvalues of the iterative matrix of the PIU iteration. The convergence of this new algorithm is also studied. The optimal iteration parameter and the corresponding optimal convergence factor for the PIU method are proposed. Moreover, we consider acceleration of the PIU iteraton by Krylov subspace methods such as GMRES method. Numerical are presented to illustrate the effective of our new method.

The paper is organized as follows. In Section 2, we introduce the parameterized inexact Uzawa method and establish the accelerated PIU model for the Eq. (1.1), then the APIU algorithm is presented. Moreover, analysis of the convergence property of this new method is given in Section 3. In Section 4, some numerical experiments are given to show the efficiency of the APIU method. Finally, some conclusion remarks are proposed in Section 5.

The following notations will be used throughout this paper. We denote the identity matrix and the 0-matrix by I and O, respectively. For a matrix $C\in R^{m\times n}$, we denote the transpose of C by $C^\top$, and the rank of matrix C is denoted as rank(C). Moreover, the spectral radius of C is denoted by $\rho(C)$. $\|\cdot\|_2$ denotes the l_2 norm of the corresponding vector.

2. The APIU method

Let $x = y + iz$ and $b = p + iq$, then from (1.1) and (1.2), we can get $(W + iT)(y + iz) = p + iq$, which implies that we can obtain the following block two-by-two systems of linear equation [13]

$$Du = \begin{pmatrix} W & -T \\ T & W \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} = g. \tag{2.1}$$

Conversely, from the linear Eq. (2.1), we can get the complex symmetric linear system (1.1). So the complex symmetric linear system (1.1) is formally identical to the above block two-by-two systems of linear Eq. (2.1). Moreover, the block two-by-two systems of linear Eq. (2.1) can be formally regarded as a special case of the generalized saddle point problem [22–25]. It frequently arises from finite element discretizations of elliptic partial differential equation (PDE)-constrained optimization problems such as distributed control problems [28–32] and so on.

Based on the parameterized inexact Uzawa (PIU) iteration method [32,33] for solving the following generalized saddle point problem

$$\begin{pmatrix} A & B \\ B^\top & -C \end{pmatrix}\begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} = \begin{pmatrix} \widetilde{p} \\ \widetilde{q} \end{pmatrix}, \tag{2.2}$$

in this section, we derive an accelerated PIU iteration method for solving the block two-by-two system of linear Eq. (2.1). To this end, we first introduce the PIU iteration method proposed in Bai and Wang [33] for Eq. (2.2). And this PIU iteration method is algorithmically described in the following.

Method 2.1. Given initial vectors $\widetilde{y}^{(0)} \in R^n$ and $\widetilde{z}^{(0)} \in R^n$ and two relaxation factors ω, τ with $\omega, \tau \neq 0$. For $k = 0, 1, 2, \ldots,$ until the iteration sequence $\left\{(\widetilde{y}^{(k)^\top}, \widetilde{z}^{(k)^\top})^\top\right\}$ converges to the exact solution of the saddle point problem (2.2), compute

$$\begin{cases} \widetilde{y}^{(k+1)} = \widetilde{y}^{(k)} + \omega P^{-1}\left(\widetilde{p} - A\widetilde{y}^{(k)} - B\widetilde{z}^{(k)}\right), \\ \widetilde{z}^{(k+1)} = \widetilde{z}^{(k)} + \tau Q^{-1}\left(B^\top\widetilde{y}^{(k+1)} - C\widetilde{z}^{(k)} - \widetilde{q}\right), \end{cases}$$

where $P\in R^{n\times n}$ and $Q\in R^{n\times n}$ are prescribed symmetric positive definite matrices.

If $P\neq A$ and $\omega = \tau = 1$ then the PIU iteration Method 2.1 yields inexact Uzawa algorithm [34–36] for solving the saddle point problems. Based on the above Method 2.1 [33], we can get the following iteration algorithm.