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Contour integration solution for a thermoelastic problem of a spherical cavity

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A B S T R A C T

We study a 1D problem of a spherical cavity whose surface is traction free and kept at a temperature that depends on the time. Laplace transform techniques are utilized. We use contour integration and the complex inversion formula to get the inverse transforms as definite integrals. Numerical computations are illustrated graphically.

1. Introduction

The theory of generalized thermoelasticity with one relaxation time proposed by Lord and Shulman [1] arose as a result of a modification of the equation of heat conduction, originally introduced by Maxwell [2] in the context of the theory of gases, and later by Cattaneo [3] in the context of heat conduction in rigid bodies. Dhaliwal and Sherief completed the derivation by extending it to the case of an anisotropic medium in [4]. This Theory was extended to micropolar materials in [5], viscoelastic materials in [6] and poroelastic materials in [7].

Linear problems of thermoelasticity are often solved by utilizing Laplace transform techniques. In most problems, exact inversion of the transform is not possible. Many writers use numerical techniques to invert the Laplace transform as in [8–16]. The finite difference method was used in [17]. Asymptotic expansions valid for small values of time are also used as in [6,18–21]. Series solutions are also used for this purpose as in [22,23]. At last, the complex inversion method is used to invert the Laplace transform to get the solution in the form of definite integrals. This method was proposed for thermoelastic problems by Sherief and Saleh [24]. The last method is the only one that gives exact solutions for problems of generalized thermoelasticity.

2. Formulation of the problem

Let (r, ϑ, φ) be a spherical coordinate system. We shall consider an isotropic, homogeneous thermoelastic body occupying the infinite region $L \le r < \infty$, where "L" is the radius of the spherical cavity. We note that due to spherical symmetry the only non-vanishing displacement component is the radial one $u_r = u(r,\ t)$.

The components of the strain tensor have the following forms [1]:

$$e_{rr} = \frac{\partial u}{\partial r},\ e_{\vartheta\vartheta} = e_{\phi\phi} = \frac{u}{r},\ e_{r\phi} = e_{r\vartheta} = e_{\vartheta\phi} = 0. \tag{1}$$

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The cubical dilatation e is thus given by

$$e = \frac{\partial u}{\partial r} + \frac{2u}{r} = \frac{1}{r^2}\frac{\partial \left(r^2 u\right)}{\partial r}. \tag{2}$$

The equation of motion when the body forces are absent, is given in vector form by [1]

$$\mu\,\nabla^2\underline{\mathbf{u}} + (\lambda+\mu)\,grad(div\,\underline{\mathbf{u}}) - \gamma\,grad\,T = \rho\ddot{\underline{\mathbf{u}}}, \tag{3}$$

where ρ is the density, T is the absolute temperature, λ, μ are Lamé's modulii and γ is a material parameter given by $\gamma=(3\lambda+2\mu)\alpha_t$ where α_t is the coefficient of linear thermal expansion. In Eq. (3), a dot denotes derivative with respect to time t. Bold letters indicate vector quantities.

The energy equation has the form [1]

$$k\,\nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_0\,\frac{\partial^2}{\partial t^2}\right)[\rho\,c_E T + \gamma\,T_0\,div\,\underline{\mathbf{u}}], \tag{4}$$

where T_0 is a fixed reference temperature, C_E is the specific heat per unit mass at constant strain, τ_0 is the relaxation time and k is the thermal conductivity. In the above equation, ∇^2 is the one dimensional Laplace operator in spherical polar coordinates, given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.$$

The constitutive equations for the components of the stress tensor can be written as [1]

$$\sigma_{rr} = 2\,\mu\,e_{rr} + \lambda\,e - \gamma\,(T - T_0), \tag{5a}$$

$$\sigma_{\vartheta\vartheta} = \sigma_{\phi\phi} = 2\mu\,e_{\vartheta\vartheta} + \lambda\,e - \gamma\,(T - T_0), \tag{5b}$$

$$\sigma_{r\vartheta} = \sigma_{r\phi} = \sigma_{\vartheta\,\phi} = 0. \tag{5c}$$

From now on, we shall use the following non-dimensional variables:

$$r' = c_1\,\eta\,r,\ u' = c_1\,\eta\,u,\ L' = c_1\,\eta\,L,\ t' = c_1^2\,\eta\,t,\ \tau_0' = c_1^2\,\eta\,\tau_0,\ \sigma_{ij}' = \frac{\sigma_{ij}}{\mu},\ \theta = \frac{\gamma\,(T - T_0)}{\lambda + 2\,\mu},$$

where $c_1 = \sqrt{\frac{\lambda+2\,\mu}{\rho}}$, $\eta = \frac{\rho\,c_E}{k}$.

In terms of these non-dimensional variables, Eqs. (3)–(5a) and (5b) take the following form (dropping the primes for convenience):

$$\nabla^2\,\underline{\mathbf{u}} + \left(\beta^2 - 1\right) grad\,div\,\underline{\mathbf{u}} - \beta^2 grad\,\theta\ = \beta^2\,\ddot{\underline{\mathbf{u}}}, \tag{6}$$

$$\nabla^2\,\theta = \left(\frac{\partial}{\partial t} + \tau_0\,\frac{\partial^2}{\partial t^2}\right)(\theta + \varepsilon\,e), \tag{7}$$

$$\sigma_{rr} = 2\,\frac{\partial u}{\partial r} + \left(\beta^2 - 2\right)e\ - \beta^2\,\theta, \tag{8a}$$

$$\sigma_{\vartheta\vartheta} = \sigma_{\phi\phi} = 2\,\frac{u}{r} + \left(\beta^2 - 2\right)e\ - \beta^2\,\theta, \tag{8b}$$

where

$$\varepsilon = \frac{T_0\,\gamma^2}{k\,\eta\,(\lambda + 2\mu)},\ \beta^2 = \frac{\lambda + 2\mu}{\mu}.$$

The boundary conditions of the problem are assumed to be

$$\sigma_{rr}(L, t) = 0, \tag{9a}$$

$$\theta(L, t) = F(t) \tag{9b}$$

$$\lim_{r\to\infty} u(r, t) = \lim_{r\to\infty}\theta(r, t) = 0 \tag{10}$$

Condition (9a) means that the surface of the cavity is traction free, i.e. there are no mechanical loads on the surface, while condition (9b) means that the surface of the cavity is kept at a known temperature which is a function of time.