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# Application of SubIval in solving initial value problems with fractional derivatives

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ABSTRACT

The application of a numerical method for the approximation of the fractional derivative (in Riemann–Liouville and Caputo definitions) in initial value problems is discussed. The method (previously known as the subinterval-based method) is now referred to by its acronym, SubIval, for simpler future references.

It is dependent on subinterval partitions (performed according to a proposed algorithm), interpolations using selected time axis nodes (i.e. nodes where solutions have been computed) and analytical monomial integrodifferentiation formulae.

Two exemplary circuit problems have been introduced as a test for the method. These problems have analytical solutions available in literature. The evaluations of these solutions have been compared with results obtained through an adaptive step size predictor-corrector scheme, where the core computations relied on the proposed numerical method.

SubIval has been implemented into an ActiveX Dynamic-Link Library (DLL). The paper contains instructions on how the predictor-corrector algorithm can be implemented into a Computer Algebra System, where the SubIval library is applied for the fractional derivative approximation. Examples of this are given in the form of scripts in MATLAB and Mathematica. These scripts generally allow to solve systems of fractional order state equations, to which the introduced circuit examples can be brought to.



## 1. Introduction

Fractional calculus is a branch introducing definitions of a fractional order derivative and integral (or generally – an integro-derivative). Various definitions can be found in literature [1], where this paper concerns the usage of the fractional derivative of order $\alpha \in (0, 1]$ in what is known as the Riemann–Liouville definition [2]:

$$ {}^{\mathrm{RL}}_{t_a}\mathrm{D}^{\alpha}_{t_b} x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_a}^{t_b} \frac{x(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau \tag{1} $$

and Caputo's definition [3]:

$$ {}^{\mathrm{C}}_{t_a}\mathrm{D}^{\alpha}_{t_b} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_a}^{t_b} \frac{x^{(1)}(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau, \tag{2} $$

where $\Gamma$ denotes the gamma function:

$$ \Gamma(z) = \int_0^{\infty} x^{z-1} \mathrm{e}^{-x}\, \mathrm{d}x. \tag{3} $$

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The difference between the above definitions for the selected interval of $\alpha$ is [4]:

$$ {}^{\mathrm{RL}}_{t_a}\mathrm{D}^{\alpha}_{t_b} x(t) = {}^{\mathrm{C}}_{t_a}\mathrm{D}^{\alpha}_{t_b} x(t) + \frac{x(t_0)}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}. \tag{4} $$

Because the paper contains often relations to intervals as symbols (e.g. for an interval $\Xi = [t_{\mathrm{start}}, t_{\mathrm{end}}]$ the symbol $\Xi$ is rather used than the variables $t_{\mathrm{start}}$ and $t_{\mathrm{end}}$) a special notation is used in the paper for both the Riemann–Liouville fractional derivative:

$$ {}^{\mathrm{RL}}\mathrm{d}^{\alpha}_{\Xi} x(t) = {}^{\mathrm{RL}}_{t_a}\mathrm{D}^{\alpha}_{t_b} x(t) \tag{5} $$

and the Caputo fractional derivative:

$$ {}^{\mathrm{C}}\mathrm{d}^{\alpha}_{\Xi} x(t) = {}^{\mathrm{C}}_{t_a}\mathrm{D}^{\alpha}_{t_b} x(t), \tag{6} $$

where $\Xi = [t_a, t_b]$. If an equation is valid for both definitions then the preceding upper index is omitted (e.g. the fractional derivative is then written as $\mathrm{d}^{\alpha}_{\Xi} x(t)$ or ${}_{t_a}\mathrm{D}^{\alpha}_{t_b} x(t)$).

Fractional calculus has many applications ranging from useful theoretical concepts to modeling. Its application can be found e.g. in:

- circuit theory (the fractional order capacitor is useful for modeling supercapacitors [5] and there have been successful attempts at modeling some coils with ferromagnetic cores by applying fractional order inductor models [6]),
- control theory and applications [7–9], where fractional PID controllers have improved stability properties,
- fractional order filter design [10,11],
- electromagnetic field analysis, when some complex materials are modeled [12,13],
- temperature field computations [14–16].

It is of course important in which context the fractional derivative appears e.g. as a derivative with respect to space or time. The presented analysis concerns only the latter case. As for how to solve problems where the fractional time derivative appears, the author can refer to, first off, analytical solutions. The obtained formulae then base on the Mittag–Leffler function [17,18]. As for other methods – here are a few often mentioned by lead researchers (e.g. in [19]):

- the Adomian decomposition method [20],
- the variational iteration method [21],
- the differential transform method [22],
- methods basing on wavelets [23,24],
- the Taylor expansion method [25],
- collocation methods [26–28],
- methods generally referred to as Fractional Linear Multistep Methods [29,30],
- methods that base on the fractional difference operator [31,32].

The analysis presented in the paper concerns the subinterval-based method (first introduced in [33]), which is now referred to by its acronym – SubIval – for a simpler reference. It is a numerical method that generally allows to approximate the fractional derivative (in both definition (1) and (2)) while solving an initial value problem. Its application results in the equation:

$$ \mathrm{d}^{\alpha}_{\Xi_{\mathrm{tot}}} x(t) \approx a x(t_{\mathrm{now}}) + b, \tag{7} $$

much like what implicit BDF (Backward Differentiation Formulae) result in for a first order derivative. The interval $\Xi_{\mathrm{tot}} = [t_0, t_{\mathrm{now}}]$ with $t_0$ being the initial time instance; $t_{\mathrm{now}}$ is a time instance for which the variable $x$ is computed (it is always the final time instance on the time axis in every step of an initial value problem solver – see Sections 2 and 3 for more details). The method relies on a partition of the integrodifferentiation interval $\Xi_{\mathrm{tot}}$ into a number of subintervals (this is explained in Section 2 along with how the relation of Eq. (7) is derived). How the setup and modification of the subintervals is performed is explained in Section 3. A thorough explanation of the SubIval basis, similar to the one given in the two upcoming sections, is also given in another paper [34]. The one here differs by an added explanation of the cooperation of SubIval with an adaptive step size predictor-corrector scheme.

## 2. Basic principles, subinterval partitions

SubIval was designed to be a variable step size method, which treats only a currently computed node implicitly, i.e. for a new time step, when a new time instance $t_j$ is selected, for a fractional derivative $\mathrm{d}^{\alpha}_{\Xi} x(t)$ only the value $x(t_j)$ is computed (this concerns all variables under a fractional derivative), so in the manner in which most popular implicit ODE solving methods treat IVPs. The newly added time value is referred to as $t_{\mathrm{now}}$. The operation $\mathrm{d}^{\alpha}_{\Xi_{\mathrm{tot}}} x(t)$ is split into a number of integrodifferentiations at designated, subsequent subintervals $\Xi_s = [t_{s,\mathrm{start}}, t_{s,\mathrm{end}}]$ (for $s > 1 : t_{s,\mathrm{start}} = t_{s-1,\mathrm{end}}$). This results in:

$$ \mathrm{d}^{\alpha}_{\Xi_{\mathrm{tot}}} x(t) = \mathrm{d}^{\alpha}_{\Xi_M} x(t) + \sum_{s=1}^{S} \mathrm{d}^{\alpha}_{\Xi_s} x(t), \tag{8} $$

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