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Dispersion analysis of triangle-based Whitney element methods for electromagnetic wave propagation

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ABSTRACT

We study the numerical dispersion/dissipation of a triangle-based edge Finite Element Method (edgeFEM) of degree $r \geq 1$ when coupled with the Leap-Frog (LF) finite difference scheme to simulate the electromagnetic wave propagation over a structured triangulation of the 2D physical domain. The analysis addresses the discrete eigenvalue problem resulting from the approximation of the dispersion relation. First, we present semi-discrete dispersion graphs by varying the approximation degree r and the number of discrete points per wavelength. The fully-discrete ones are then obtained by varying also the time step. Numerical results for the edgeFEM, resp. edgeFEM-LF, are compared with those for the node Finite Element Method (nodeFEM), resp. nodeFEM-LF, applied to the considered problem.

1. Introduction

A critical factor in the simulation of wave propagation phenomena is the artificial dispersion and dissipation inflicted on the waves by the numerical scheme. The adopted scheme produces a computed wave that can indeed look plausible, qualitatively correct, even when suffering from a phase shift or a loss in the amplitude. It is thus important to address the dispersion/dissipation analysis of any scheme adopted for the simulation of wave propagation phenomena to quantify these errors numerically. A rough way to test dispersion/dissipation for a numerical scheme consists in observing the computed wave moving in a given direction for a very long time (under the assumption that the adopted time advancing scheme is stable) in a homogeneous isotropic unbounded domain. If the computed wave presents a shift in the peaks with respect to those of the analytical one, the scheme is dispersive; if the amplitude of the computed wave decreases with respect to that of the analytical one with the time passing, the scheme is dissipative. In the present work, we rather addresses the discrete eigenvalue problem resulting from the approximation of the dispersion relation on a periodic mesh. Indeed, Fourier analysis, and as a consequence, the notion of waves and their propagation, relies on the property of invariance under the group of translations. This of course is lost by discretization, but one tries to keep as much of it as possible by working on a periodic mesh. The underlying group "tends to" the group of translations when the size h of the mesh elements tends to 0 (on this subject, see for example, [8,25]).

In the simulation of propagation phenomena, both high-order accuracy and computational efficiency are mandatory. High-order methods see *e.g.* [2,7,10], are widely used to this purpose because they are known to provide a given accuracy

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with a lower number of degrees of freedom (dofs) than low-order methods. They are also compatible with a domain decomposition framework and well-suited for parallel computations (see for example [5,9,21]). In this work, we investigate dispersive and dissipative behavior of two well-established techniques for the approximation of electromagnetic phenomena, namely, the high-order node (nodeFEM) and edge (edgeFEM) Whitney finite elements. More precisely, we consider the electromagnetic wave equation expressed in terms of the electric field vector and we discretize it in space, by the node or edge Whitney finite elements presented in [18,19] and, in time, by well-known finite difference schemes, such as the Leap-Frog (LF) described in [20]. A detailed dispersion analysis can be found either in [15,23], for low order finite element schemes for Maxwell's equations on simplicial grids, or in [1], for high-order finite element schemes on quadrilateral meshes for the second-order hyperbolic problems.

The dissipation and dispersion properties of the resulting semi-discrete edgeFEM and fully discrete edgeFEM-LF approximations in 2D are analyzed (similar steps are done in [12] for the elastodynamics case). Numerical results with edge elements are compared with those obtained for node finite elements (nodeFEM, nodeFEM-LF) applied to the same problem. The model problem and its variational formulation are presented in Section 2. To ease the presentation of the grid dispersion/dissipation analysis by the so called Von Neumann technique [11,13], we shall limit ourselves to a 2D model problem and we shall only consider triangular elements. We look first at the space discretization and then at the time one. Therefore, Section 3 is dedicated to the semi-discrete formulation by the edgeFE method followed by a "partial" dispersion/dissipation analysis in Section 4 and corresponding results in Section 5. Here the terminology "partial" refers to an exact treatment in time. In Section 6 we present the fully-discrete formulation obtained by coupling the edgeFEM with a Leap-Frog (LF) finite difference scheme for the time derivatives. The "global" dispersion/dissipation analysis is thus presented and related results are shown in Section 7. Some remarks and conclusions presented in Section 8 end the paper.

2. The model problem and its weak form

The electromagnetic wave propagation over the temporal interval $[0,\, t_f]$, with t_f real and positive, in a non-conducting region $\Omega \subset \mathbb{R}^2$ is described by the following equation[1]:

$$\varepsilon\, \partial_{tt}\mathbf{u} - \mathbf{curl}\,(\mu^{-1}\mathrm{curl}\,\mathbf{u}) = \mathbf{J}_e, \quad \text{in } \Omega \times [0, t_f], \tag{1}$$

coupled with suitable boundary conditions over $\partial\Omega \times [0,\, t_f]$, and initial conditions in $\bar{\Omega} \times \{0\}$. Here $\mathbf{u}(\mathbf{x},\, t)$ is the electric field vector in $\mathbf{x} \in \Omega$ at time t, ε (resp. μ) is the material electric permittivity (resp. magnetic permeability), $\mathbf{J}_e$ the external current source and $\mathbf{u}_0$, $\mathbf{v}_0$ assigned vector fields to set the initial conditions for $\mathbf{u}$ and $\partial_t\mathbf{u}$. The propagation wave velocity in the medium is $c = 1/\sqrt{\varepsilon\mu}$. To perform the dispersion/dissipation analysis of the approximation method, we have to simplify the physical/mathematical situation in order to eliminate any possible pollution of the numerical results from data and geometrical settings. The following assumptions are generally made whenever the Von Neumann's method (plane-wave analysis) is considered.

Assumption 1. Problem (1) has to be set in $\mathbb{R}^2 \times [0, t_f]$. This is realized here by working in a squared domain $\bar{\Omega}$, with sides in the x and y directions, and imposing on its boundary $\partial\Omega = \Gamma_{\text{right}} \cup \Gamma_{\text{left}} \cup \Gamma_{\text{up}} \cup \Gamma_{\text{bottom}}$ the periodic conditions $\mathbf{u}\cdot\mathbf{t}_{|\Gamma_{\text{right}}} = \mathbf{u}\cdot\mathbf{t}_{|\Gamma_{\text{left}}}$, $\mathbf{u}\cdot\mathbf{t}_{|\Gamma_{\text{up}}} = \mathbf{u}\cdot\mathbf{t}_{|\Gamma_{\text{bottom}}}$, during the time interval $[0,\, t_f]$, where $\mathbf{t}_{|\gamma}$ denotes the unit tangent vector to the line γ.

Assumption 2. The material in Ω is isotropic and homogeneous, that is, physical coefficients ε, μ are constant scalar functions in Ω. Here, we set ε, μ equal to one so that $c = 1$.

Assumption 3. The physical situation is source free, namely, $\mathbf{J}_e \equiv \mathbf{0}$.

To write the weak formulation of Eq. (1) (under Assumptions 1–3) that is suitable for finite element discretizations, we take the dot product of (1) with a vector test function $\mathbf{v}$, integrate over the domain Ω and use Green's identity for the term containing the curl of $\mathbf{u}$. We thus obtain: $\forall\, t \in\,]0,\, t_f]$, find $\mathbf{u} = \mathbf{u}(t) \in \mathcal{V}_P$ such that

$$d_{tt}\int_\Omega \mathbf{u}\cdot\mathbf{v} + \int_\Omega (\mathrm{curl}\,\mathbf{u})(\mathrm{curl}\,\mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathcal{V}_0, \tag{2}$$

subject to the initial conditions $\mathbf{u}(0) = \mathbf{u}_0$, $d_t\mathbf{u}(0) = \mathbf{v}_0$.

For nodal approximations, $\mathcal{V}_P = \{\mathbf{u} \in H^1(\Omega)^2,\ \mathbf{u}_{|\Gamma_{\text{right}}} = \mathbf{u}_{|\Gamma_{\text{left}}},\ \mathbf{u}_{|\Gamma_{\text{up}}} = \mathbf{u}_{|\Gamma_{\text{bottom}}}\}$, with $H^1(\Omega) = \{v \in L^2(\Omega),\, \mathbf{grad}\, v \in L^2(\Omega)^2\}$, and $\mathcal{V}_0 = \{\mathbf{v} \in \mathcal{V},\, (\mathbf{v})_{|\partial\Omega} = \mathbf{0}\}$. Problem (2) in the nodal case admits a unique solution $\mathbf{u} \in \mathcal{C}^0(]0, t_f]; \mathcal{V}_P) \cap \mathcal{C}^1(]0, t_f]; L^2(\Omega)^2)$ satisfying suitable stability estimates (see A.6 in [22] for the properties of the curlcurl operator and Section 14.5 in [17]), provided that the initial conditions $\mathbf{u}_0 \in \mathcal{V}_P$, $\mathbf{v}_0 \in L^2(\Omega)^2$.

[1] In two spatial dimensions, say x, y, we denote by ∂_x (resp. ∂_y) the first order operator that associates with any differentiable scalar function g its partial derivative $\partial_x g$ (resp. $\partial_y g$) with respect to the variable x (resp. y). For any vector field $\mathbf{u} \in \mathbb{R}^2$, with $\mathbf{u} = (u_x, u_y)^\top$, we define $\mathrm{curl}\,\mathbf{u}$ as the scalar function $\partial_x u_y - \partial_y u_x$; for any scalar function $v \in \mathbb{R}$, we define $\mathbf{curl}\, v$ as the vector field $(\partial_y v, -\partial_x v)^\top$. Note that $\mathbf{curl}\, v \perp \mathbf{grad}\, v$, where $\mathbf{grad}\, v$ is the vector field $(\partial_x v, \partial_y v)^\top$. For any vector field $\mathbf{u} \in \mathbb{R}^2$, we denote by $\partial_t\mathbf{u}$ (resp. $\partial_{tt}\mathbf{u}$) the vector $(\partial_t u_x, \partial_t u_y)^\top$ (resp. $(\partial_{tt} u_x, \partial_{tt} u_y)^\top$) where ∂_t (resp. ∂_{tt}) stands for the first order (resp. second order) time derivative operator.

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