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Some splines produced by smooth interpolation

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The spline theory can be derived from two sources: the algebraic one (where splines are understood as piecewise smooth functions satisfying some continuity conditions) and the variational one (where splines are obtained via minimization of some quadratic functionals with constraints). We show that the general variational approach called *smooth interpolation* introduced by Talmi and Gilat covers not only the cubic spline but also the tension spline (called also spline in tension or spline with tension) in one or more dimensions. We show the results of a 1D numerical example that present the advantages and drawbacks of the tension spline.

1. Introduction

In most 1D practical cases, the minimum curvature (or cubic spline) method produces a visually pleasing smooth interpolating curve or surface. In some cases, however, this approach can create strong artificial oscillations. A general remedy suggested e.g. by Schweikert [7] is known as *tension spline*. The functional minimized is the weighted sum of the first derivative term and the second derivative term.

Smooth approximation [11] is a method for data interpolating or data fitting that uses the variational formulation of the problem in a normed space with constraints that represent the approximation conditions. The cubic spline interpolation in 1D is an approximation of this kind, too.

Smooth approximation can be very useful in solving problems of interpolation or data fitting, in particular in 2D and 3D. Recently, the range of its application has been extended to problems of computer aided geometric design, i.e. the mathematical description of shape for use in computer graphics, manufacturing, or analysis. Moreover, smooth approximation is widely used in constructing geographic information systems that are designed to capture, store, manipulate, analyze, manage and present geographic data or spatial data in general.

For the cubic spline, the mathematical problem is to minimize the L^2 norm of second derivative of the approximating function. Then a more sophisticated criterion is to minimize, with some weights chosen, the integrals of the squared magnitude of some (or possibly all) derivatives of a sufficiently smooth approximating function. In this paper, we construct

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the tension spline by means of the smooth approximation theory (cf. [6]), i.e. including the exact interpolation of the data at nodes and, simultaneously, with the smoothness of the interpolating curve and its first derivative.

A general approach to abstract splines including their variational definition can be found e.g. in [5] containing a lot of further references.

Most results are presented and proven for a 1D domain. Their generalization to higher dimension usually follows. Using the approach of [8,11], we introduce the problem to be solved in Section 3. We also present the general existence theorem for smooth interpolation proven in [8]. We are concerned with the use of basis system $\exp(ikx)$ of exponential functions of pure imaginary argument for 1D, 2D, and 3D smooth interpolation problems in Sections 4–7. This basis system, in particular, provides splines. In Section 7 we study some properties of this basis system suitable for measuring the smoothness of the interpolation and for generating the tension spline. We also show results of a 1D numerical experiment in Section 8 and discuss them briefly to illustrate the properties of smooth interpolation.

2. General problem of data interpolation

Basic notation and fundamental statements are presented, e.g., in [9]. Consider a finite number N of (complex, in general) measured (sampled) values $f_1, f_2, \dots, f_N \in C$ obtained at N nodes $X_1, X_2, \dots, X_N \in R^n$. The nodes are assumed to be mutually distinct. We are usually interested also in the unknown values corresponding to points in some domain. Assume that $f_j = f(X_j)$ are measured values of some continuous function f and z is an approximating function to be constructed. The dimension n of the independent variable can be arbitrary.

Definition 1 (Interpolation)**.** The *interpolating function* (interpolant) z is constructed to fulfill the interpolation conditions

$$z(X_j) = f_j, \quad j = 1, \dots, N. \tag{1}$$

Various additional conditions can be considered, e.g. minimization of some functionals applied to z.

The problem of data interpolation does not have a unique solution. The property (1) of the interpolating function is uniquely formulated by mathematical means but there are also requirements on the *subjective perception* of the behavior of the approximating curve or surface between nodes that cannot be formalized easily [12].

The general *problem of smooth approximation* (*smooth curve fitting, data smoothing*), where the interpolation condition (1) is not applied, is treated in more detail e.g. in [8,11].

For the sake of simplicity we now put $n = 1$ and assume that $X_1, X_2, \dots, X_N \in \Omega$, where either $\Omega = [a, b]$ is a finite interval or $\Omega = (-\infty, \infty)$. We will turn back to general $n \geq 1$ in Section 6.

3. Smooth interpolation according to [11]

We introduce an inner product space to formulate the additional constraints in the problem of smooth approximation [9,11]. Let $\widetilde{\mathcal{W}}$ be a linear vector space of complex valued functions g continuous together with their derivatives of all orders on the interval Ω. Let $\{B_l\}_{l=0}^{\infty}$ be a sequence of nonnegative numbers and L the smallest nonnegative integer such that $B_L > 0$ while $B_l = 0$ for $l < L$. For $g, h \in \widetilde{\mathcal{W}}$, put

$$(g, h)_L = \sum_{l=0}^{\infty} B_l \int_{\Omega} g^{(l)}(x)[h^{(l)}(x)]^* \,\mathrm{d}x, \tag{2}$$

$$|g|_L^2 = \sum_{l=0}^{\infty} B_l \int_{\Omega} |g^{(l)}(x)|^2 \,\mathrm{d}x, \tag{3}$$

where * denotes the complex conjugate.

If $L = 0$ (i.e. $B_0 > 0$), consider functions $g, h \in \widetilde{\mathcal{W}}$ such that the values of $|g|_0$ and $|h|_0$ exist and are finite. Then $(g, h)_0 = (g, h)$ has the properties of *inner product* and the expression $|g|_0 = \|g\|$ is *norm* in a normed space $W_0 = \widetilde{\mathcal{W}}$.

Let $L > 0$. Consider again functions $g \in \widetilde{\mathcal{W}}$ such that the value of $|g|_L$ exists and is finite. Let $P_{L-1} \subset \widetilde{\mathcal{W}}$ be the subspace whose basis $\{\varphi_p\}$ consists of monomials

$$\varphi_p(x) = x^p, \quad p = 0, \dots, L-1.$$

Then $(\varphi_p, \varphi_q)_L = 0$ and $|\varphi_p|_L = 0$ for $p, q = 0, \dots, L-1$. Using (2) and (3), we construct the *quotient space* $\widetilde{\mathcal{W}}/P_{L-1}$ whose zero class is the subspace P_{L-1}. Finally, considering $(\cdot, \cdot)_L$ and $|\cdot|_L$ in every equivalence class, we see that they represent the inner product and norm in the normed space $W_L = \widetilde{\mathcal{W}}/P_{L-1}$ [9].

W_L is the normed space where we minimize functionals and measure the smoothness of the interpolation. For an arbitrary $L \geq 0$, choose a *basis system* of functions $\{g_k\} \subset W_L$, $k = 1, 2, \dots$, that is complete and orthogonal (in the inner product in W_L), i.e.,

$$(g_k, g_m)_L = 0 \quad \text{for} \quad k \neq m,$$
$$(g_k, g_k)_L = |g_k|_L^2 > 0.$$