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Subclass of m-quasiconformal harmonic functions in association with Janowski starlike functions

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ABSTRACT

Let's take $f(z) = h(z) + \overline{g(z)}$ which is an univalent sense-preserving harmonic functions in open unit disc $\mathbb{D} = \{z : |z| < 1\}$. If $f(z)$ fulfills $|w(z)| = |\frac{g'(z)}{h'(z)}| < m$, where $0 \le m < 1$, then $f(z)$ is known m-quasiconformal harmonic function in the unit disc (Kalaj, 2010) [8]. This class is represented by $S_{H(m)}$.

The goal of this study is to introduce certain features of the solution for non-linear partial differential equation $\overline{f_{\bar{z}}} = w(z)f(z)$ when $|w(z)| < m$, $w(z) \prec \frac{m^2(b_1 - z)}{m^2 - \overline{b_1} z}$, $h(z) \in S^*(A, B)$. In such case $S^*(A, B)$ is known to be the class for Janowski starlike functions. We will investigate growth theorems, distortion theorems, jacobian bounds and coefficient ineqaulities, convex combination and convolution properties for this subclass.

1. Introduction

Let's take Ω which is a family of regular functions $\phi(z)$ in the disc $\mathbb{D}$ and fulfilling $\phi(0) = 0$, $|\phi(z)| < 1$ for each $z \in \mathbb{D}$.

Then, the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ (which is regular in $\mathbb{D}$) is shown by $P(A, B)$ for arbitrary fixed real numbers A, B, $-1 \le B < A \le 1$. In this condition $p(z)$ is in $P(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \tag{1.1}$$

for certain $\phi(z)$ is an element of Ω and each z is in $\mathbb{D}$.

Furthermore, lets take $S^*(A, B)$ indicating the class of regular functions $s(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ is an element of $\mathbb{D}$ and $s(z) \in S^*(A, B)$ if and only if

$$z\frac{s'(z)}{s(z)} = p(z) \tag{1.2}$$

for certain $p(z)$ is an element of $P(A, B)$ and each z is in $\mathbb{D}$ [7]. Let's take $s_1(z) = z + d_2 z^2 + \cdots$ and $s_2(z) = z + e_2 z^2 + \cdots$ which are the family of analytic functions belong to $\mathbb{D}$. If a function of $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ exists for each $z \in \mathbb{D}$, then Subordination and Lindelöf principle [3,5] implies that $s_1(z)$ is subordinate to $s_2(z)$ and $s_1(z) \prec s_2(z)$ can be written if and only if $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$, $s_1(0) = s_2(0)$ and $s_1(\mathbb{D}_r) \subset s_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z : |z| < r, 0 < r < 1\}$.

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Ultimately, a harmonic mapping in $\mathbb{D}$, which is a complex-valued harmonic function f, transforms $\mathbb{D}$ onto the domain $f(\mathbb{D})$. The mapping f is written as $f(z) = h(z) + \overline{g(z)}$ and $\mathbb{D}$ is a simply-connected domain. The sum of $h(z)$ and $g(z)$, which are analytic in $\mathbb{D}$, is known as canonical representation. They have the expansions of power series given below

$$h(z) = \sum_{m=0}^{\infty} a_m z^m, \quad g(z) = \sum_{m=0}^{\infty} b_m z^m \tag{1.3}$$

where $a_n, b_n \in C$, $n = 0, 1, 2, 3, \ldots$. In this case $h(z)$ and $g(z)$ are the analytic and co-analytic parts of $f(z)$, respectively. Duren's monograph in [4] showed that the following equation is correct for relation between Jacobian and local univalent of $f(z)$.

$$J_f = |h'(z)|^2 - |g'(z)|^2 \neq 0 \Leftrightarrow f(z). \tag{1.4}$$

This result reveals that, if the conditions of $|g'(z)|>|h'(z)|$ or $|g'(z)|<|h'(z)|$ hold in $\mathbb{D}$, the locally univalent harmonic mappings are known to be sense-preserving or sense-reserving in $\mathbb{D}$ respectively [12].

This paper is restricted to sense-preserving harmonic mappings. It is remarkable to note that the canonical representation of $f(z)$ is sense-preserving if and only if $h'(z)$ doesn't cancel out in $\mathbb{D}$. $w(z) = \frac{g'(z)}{h'(z)}$, which is known second dilatation, satisfies the inequality of $|w(z)| < 1$ for each $z \in \mathbb{D}$. As a result, each sense-preserving harmonic mapping class in $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ are shown by S_H. Hence S_H covers the the class of S which is univalent. The family of each mappings $f \in S_H$, which has the condition, $g'(0) = 0$, i.e, $b_1 = 0$ is represented by S_H^0. Thus, it can be easily seen that $S \subset S_H^0 \subset S_H$. The lemma and theorem given below are required for the goal of this study.

Lemma 1.1 ([6]). *Let's take a function of $\phi(z)$ which is non-constant in $\mathbb{D}$ and under the condition of $\phi(0) = 0$. If $|\phi(z)|$ reaches to the highest value of its at $|z| = r$ and z_0, then $z_0\phi'(z_0) = k\phi(z_0)$, $k \geq 1$.*

Theorem 1.2 ([7]). *If $s(z)$ is an element of $S^*(A, B)$, then the following equations can be written for $|z| = r$ and $0 < r < 1$.*

$$F(r, -A, -B) \leq |s(z)| \leq F(r, A, B) \tag{1.5}$$

$$F(r, A, B) = \begin{cases} r(1 + Br)^{\frac{A-B}{B}} & \text{for } B \neq 0, \\ re^{Ar} & \text{for } B = 0. \end{cases} \tag{1.6}$$

These sharp bounds are obtained in $z = re^{i\theta}$ where θ changes from zero to 2π for

$$s(z) = \begin{cases} z(1 + Be^{-i\theta}z)^{\frac{A-B}{B}} & \text{for } B \neq 0, \\ ze^{Ae^{-i\theta}z} & \text{for } B = 0. \end{cases} \tag{1.7}$$

2. Main results

Lemma 2.1. *Let $p(z)$ is an element of $P(A, B)$. If*

$$z\frac{s'(z)}{s(z)} = p(z) = (A + iB) + p_1 z + p_2 z^2 + \cdots . \tag{2.1}$$

be analytic in $\mathbb{D}$ and satisfies the condition $Rep(z) > 0$ then

$$\frac{s(z)M(A, B, -r)}{z(1 - r^2)} \leq s'(z) \leq \frac{s(z)M(A, B, r)}{z(1 - r^2)} \tag{2.2}$$

where $M(A, B, r) = \frac{2Ar + (A^2 + (1 + A^2) + B^2(1 - r^2)^2)}{1 - r^2}$.

Proof. Let $p(z) = (A + iB) + p_1(z) + p_2 z^2 + \cdots$ is analytic in the open unit disc $\mathbb{D}$ and satisfies the condition $Rep(z) > 0$ then the function

$$p_1(z) = \frac{1}{A}(p(z) - iB)$$

is in P. (See [9]). On the other hand if $p_1(z)$ is element of P, then we have

$$|p_1(z) - \frac{1 + r^2}{1 - r^2}| \leq \frac{2r}{1 - r^2} \tag{2.3}$$

After algebraic calculation we get the result. □

Lemma 2.2. *Let's consider $f(z) = h(z) + \overline{g(z)} \in S_H$ and $h(z) \in S^*(A, B)$, then*

$$\frac{g(z)}{h(z)} = \frac{m^2(b_1 - \phi(z))}{m^2 - \overline{b_1}\phi(z)} \quad \text{where } \phi(z) \in \Omega \text{ and } 0 \leq m < 1.$$