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Bracketing the solutions of an ordinary differential equation with uncertain initial conditions

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ABSTRACT

In this paper, we present a new method for bracketing (*i.e.*, characterizing from inside and from outside) all solutions of an ordinary differential equation in the case where the initial time is inside an interval and the initial state is inside a box. The principle of the approach is to cast the problem into bracketing the largest positive invariant set which is included inside a given set $\mathbb{X}$. Although there exists an efficient algorithm to solve this problem when $\mathbb{X}$ is bounded, we need to adapt it to deal with cases where $\mathbb{X}$ is unbounded.

1. Introduction

In this paper, we deal with a dynamical system $\mathcal{S}$ defined by the following state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector and $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the evolution function of $\mathcal{S}$. Denote by $\boldsymbol{\varphi}_\mathbf{f}$ the flow map of the system. This means that if at time t_0, the initial state vector is $\mathbf{x}_0$, then the solution of the state equation is

$$\mathbf{x}(t) = \boldsymbol{\varphi}_\mathbf{f}(t - t_0, \mathbf{x}_0). \tag{2}$$

In this paper, we consider that the initial state $\mathbf{x}_0$ is not known exactly. More precisely, $\mathbf{x}_0$ belongs to a box $[\mathbf{x}_0]$ of $\mathbb{R}^n$. Two problems will be treated.

Problem 1 (Forward reachable set)**.** We define the *forward reachable set* [2,16,21] as

$$\mathbb{F}^+_{[\mathbf{x}_0]} = \{\mathbf{x}_a \mid \exists \mathbf{x}_0 \in [\mathbf{x}_0], \exists t \geq 0, \mathbf{x}_a = \boldsymbol{\varphi}_\mathbf{f}(t, \mathbf{x}_0)\}. \tag{3}$$

The problem that we will consider is to bracket the set $\mathbb{F}^+_{[\mathbf{x}_0]}$ which means that we want to characterize this set from inside and from outside. This set can be interpreted as an approximation of all solutions of (1), except that we loose the dependency with respect to t.

Problem 2 (Positive graph)**.** For a fixed, t_0 and $\mathbf{x}_0$, the positive graph of the solution of (1) corresponds to the set [3]

$$\mathbb{G}^+_{t_0,\mathbf{x}_0} = \{(t, \mathbf{x}_a) \mid t \geq t_0, \mathbf{x}_a = \boldsymbol{\varphi}_\mathbf{f}(t - t_0, \mathbf{x}_0)\}. \tag{4}$$

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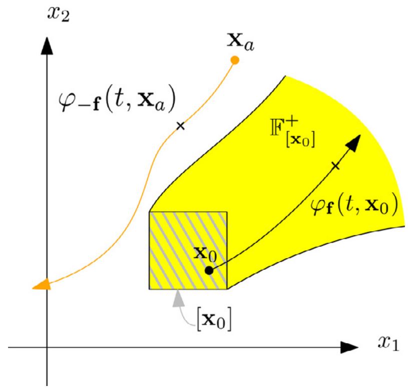

Fig. 1. Illustration of the set $\mathbb{F}^+_{[\mathbf{x}_0]}$. The orange backward trajectory is outside $\mathbb{F}^+_{[\mathbf{x}_0]}$ since it never reaches $[\mathbf{x}_0]$. The black trajectory is included in $\mathbb{F}^+_{[\mathbf{x}_0]}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

We still assume that $\mathbf{x}_0 \in [\mathbf{x}_0]$ but also, we consider that the initial time t_0 is uncertain and is only known to belong to the interval $[t_0]$. In this context, we define the *positive graph* as the set

$$\mathbb{G}^+_{[t_0],[\mathbf{x}_0]} = \{(t, \mathbf{x}_a) | \exists t_0 \in [t_0], \exists \mathbf{x}_0 \in [\mathbf{x}_0] \mid t \geq t_0, \mathbf{x}_a = \boldsymbol{\varphi}_{\mathbf{f}}(t - t_0, \mathbf{x}_0)\}, \tag{5}$$

which can be interpreted as the solution of the state equation with uncertain initial state and time. Similarly to Problem 1, we want to bracket the set $\mathbb{G}^+_{[t_0],[\mathbf{x}_0]}$ from inside and outside.

Our objective is to find a unique algorithm able to find a guaranteed inner and outer approximation of the sets $\mathbb{F}^+_{[\mathbf{x}_0]}$ and $\mathbb{G}^+_{[t_0],[\mathbf{x}_0]}$ [9]. Some existing approaches use guaranteed integration [6,20,23] to bracket those sets [7]. For efficiency reasons we will propose in this paper, a guaranteed approach based on *interval computation* [9,13] and *constraint networking* [14] that do not use guaranteed integration. The main difference with existing approaches is that bisections will take place both in the time space and the state space, which makes the method both Eulerian and Lagrangian [15]. This increases the complexity of the method but allows us to have a better control on the accuracy of the results.

2. Main results

This section shows that both problems proposed in Section 1 can be expressed as the computation of the largest positive invariant set [10] which is included inside a given set $\mathbb{X}$.

A set $\mathbb{A}$ is *positive invariant* for the system (1) if for any trajectory $\mathbf{x}(\cdot)$, we have

$$\mathbf{x}(0) \in \mathbb{A}, t \geq 0 \implies \mathbf{x}(t) \in \mathbb{A}. \tag{6}$$

Given a set $\mathbb{X}$, we denote by $Inv^+(\mathbf{f}, \mathbb{X})$, the largest subset of $\mathbb{X}$ (with respect to the inclusion) which is positive invariant. The *largest* set exists and is unique, due to the fact that the set of positive invariant sets is a complete lattice with respect to the inclusion (*e.g.*, the union or the intersection between two positive invariant sets is positive invariant). From [3], we know that

$$\mathbf{x}_a \in Inv^+(\mathbf{f}, \mathbb{X}) \Leftrightarrow \boldsymbol{\varphi}_{\mathbf{f}}([0, \infty], \mathbf{x}_a) \subset \mathbb{X}. \tag{7}$$

As a consequence, $Inv^+(\mathbf{f}, \mathbb{X})$ can be defined in two different manners

$$\begin{aligned} Inv^+(\mathbf{f}, \mathbb{X}) &= \bigcup \{\mathbb{A} \in \mathcal{P}(\mathbb{X}) | \mathbb{A} \text{ is positive invariant}\} \\ &= \{\mathbf{x}_a \in \mathbb{R}^n \mid \boldsymbol{\varphi}_{\mathbf{f}}([0, \infty], \mathbf{x}_a) \subset \mathbb{X}\} \end{aligned} \tag{8}$$

where $\mathcal{P}(\mathbb{X})$ is the power set of $\mathbb{X}$.

The two following theorems show that our two sets $\mathbb{F}^+_{[\mathbf{x}_0]}$ and $\mathbb{G}^+_{[t_0],[\mathbf{x}_0]}$ can be defined in terms of positive invariant sets.

Theorem 1. *We have*

$$\mathbb{F}^+_{[\mathbf{x}_0]} = \mathbb{R}^n \backslash Inv^+(-\mathbf{f}, \mathbb{R}^n \backslash [\mathbf{x}_0]) \tag{9}$$

where $\backslash$ *is the set theoretic difference operator (i.e.,* $\mathbb{A} \backslash \mathbb{B} = \{\mathbf{x} \in \mathbb{A} | \mathbf{x} \notin \mathbb{B}\}$*) and* $-\mathbf{f}$ *is the opposite of* $\mathbf{f}$ *(i.e.,* $\forall \mathbf{x}, -\mathbf{f}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \mathbf{0}$*).*

Proof. Take an element $\mathbf{x}_a$ of $Inv^+(-\mathbf{f}, \mathbb{R}^n \backslash [\mathbf{x}_0])$, as illustrated by Fig. 1, we have

$$\begin{aligned} & \mathbf{x}_a \in Inv^+(-\mathbf{f}, \mathbb{R}^n \backslash [\mathbf{x}_0]) \\ \Leftrightarrow \quad & \boldsymbol{\varphi}_{-\mathbf{f}}([0, \infty], \mathbf{x}_a) \subset \mathbb{R}^n \backslash [\mathbf{x}_0] \end{aligned}$$