ARTICLE IN PRESS
JID: AMC [m3Gsc;July 18, 2017;15:57]

Applied Mathematics and Computation 000 (2017) 1–13

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An a posteriori estimator of eigenvalue/eigenvector error for penalty-type discontinuous Galerkin methods

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ARTICLE INFO

Article history:
Available online xxx

MSC:
Primary: 65N30
Secondary: 65N25
65N15

Keywords:
Eigenvalue problem
Finite element method
A posteriori error estimates
Discontinuous Galerkin method

ABSTRACT

We provide an abstract framework for analyzing discretization error for eigenvalue problems discretized by discontinuous Galerkin methods such as the local discontinuous Galerkin method and symmetric interior penalty discontinuous Galerkin method. The analysis applies to clusters of eigenvalues that may include degenerate eigenvalues. We use asymptotic perturbation theory for linear operators to analyze the dependence of eigenvalues and eigenspaces on the penalty parameter. We first formulate the DG method in the framework of quadratic forms and construct a companion infinite dimensional eigenvalue problem. With the use of the companion problem, the eigenvalue/vector error is estimated as a sum of two components. The first component can be viewed as a “non-conformity” error that we argue can be neglected in practical estimates by properly choosing the penalty parameter. The second component is estimated a posteriori using auxiliary subspace techniques, and this constitutes the practical estimate.

1. Introduction

We present an a posteriori error analysis for penalty type Discontinuous Galerkin (DG) methods. All such numerical methods have in common the presence of a penalty term that ensures the stability of the methods. The function of the penalty term is to control the magnitude of the jumps of the discontinuous solution across the faces of the mesh [4,31,33]. An example of penalty term is the bilinear form $J(\cdot, \cdot)$ defined below. The strength of the penalization delivered by penalty terms can be adjusted using a parameter denoted in this work with τ. In general, there is not a unique way to choose the value of τ, but the analysis can prescribe how an appropriate value can be chosen to ensure stability; see, for example, Proposition 3.6, which shows that the penalty term τ has to be sufficiently large for the symmetric interior penalty method. When penalty terms are applied to faces of the mesh along the boundary of the domain, their function becomes to enforce boundary conditions weakly. This is natural in the context of DG methods, but this practice can be traced back to continuous Galerkin methods [32].

Motivated by the approach from [13], we introduce a notion of the companion discontinuous Galerkin forms. We present estimates for both multiple and clustered eigenvalues and also provide estimates for the associated invariant subspaces. In contrast to [13], our approach is based on the theory of the monotone convergence for quadratic forms from [18,27]. This

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http://dx.doi.org/10.1016/j.amc.2017.07.007
0096-3003/

Please cite this article as: S. Giani et al., An a posteriori estimator of eigenvalue/eigenvector error for penalty-type discontinuous Galerkin methods, Applied Mathematics and Computation (2017), http://dx.doi.org/10.1016/j.amc.2017.07.007

theory has been adapted to the application in numerical analysis in [14]. In the present paper we apply the abstract results from [14] to split the approximation error for the eigenvalues and spectral projections into the nonconformity estimate and the a posteriori computable error estimator.

Although our analysis is based on the abstract operator theory from [14] and our results directly include more general second order differential operators in the div − grad form, we will concentrate our presentation on the Laplace eigenvalue problem as a prototypical elliptic eigenvalue problem.

Let us make this claim more plausible. Assume we are given a positive-definite family of forms

$$\mathcal{B}_\tau(u,v) = B(u,v) + \tau J(u,v), \qquad u,v \in V \subset \mathcal{H} \ . \tag{1}$$

It is assumed that $\mathcal{B}_\tau$ are closed and densely defined in $\mathcal{H}$ and that the form J is positive semidefinite and bounded on V. An example of $\mathcal{B}_\tau$ which satisfies these requirements is immediately provided by the Symmetric Interior Penalty Discontinuous Galerkin (SIPDG) [4] and Local Discontinuous Galerkin (LDG) [19] discretizations of the Laplace operator in a bounded polygonal domain Ω.

To be precise, let $\mathcal{T}$ be a triangulation of Ω and let the functions $\underline{p}$ and h be the degree distribution function and the element diameter function, respectively. Here and in what follows, we will consider the piecewise polynomial space $S_{\underline{p}}(\mathcal{T})$, which is defined by requiring that each $u \in S_{\underline{p}}(\mathcal{T})$ is such that each restriction $.u|_K$ is a polynomial of degree at most $\underline{p}(K)$ for each element $K \in \mathcal{T}$. We now define the discontinuous Galerkin space $V = S_{\underline{p}}(\mathcal{T}) + H^1_0(\Omega)$ and recall the notation $\{\{\cdot\}\}$ and $[[\cdot]]$ for standard jump operators and the appropriate lifting operator $\mathcal{L}$ for the discrete gradients. We now introduce the bilinear forms

$$B(u,v) = \sum_{K\in\mathcal{T}} \int_K (\nabla u - \mathcal{L}([\![u]\!])) \cdot (\nabla v - \mathcal{L}([\![v]\!])) \ ,$$

$$J(u,v) = \sum_{F\in\mathcal{F}(\mathcal{T})} \frac{p_F^2}{h_F} \int_F [\![u]\!] \cdot [\![v]\!] \ .$$

It is a standard result on the local discontinuous Galerkin method that the forms B and J satisfy the requirements of (1), cf. [4,19] and Section 3. We call this $\mathcal{B}_\tau$ the companion form of the Laplace eigenvalue problem

$$-\Delta u = \lambda u \text{ in } \Omega \quad , \quad u = 0 \text{ on } \partial\Omega \ . \tag{2}$$

This is further justified if we observe that the variational formulation of (2)

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda (u,v) \qquad \forall\, v \in H^1_0(\Omega) \ , \tag{3}$$

where $(\cdot,\cdot)$ is the L^2 inner-product and $\|\cdot\|$ the L^2 norm, can be obtained by restricting the form $\mathcal{B}_\tau$ to the subspace $H^1_0(\Omega) \subset V$. On the other hand the discrete eigenvalue problem

$$\sum_{K\in\mathcal{T}} \int_K (\nabla u - \mathcal{L}([\![u]\!])) \cdot (\nabla v - \mathcal{L}([\![v]\!])) + \tau \sum_{F\in\mathcal{F}(\mathcal{T})} \frac{p_F^2}{h_F} \int_F [\![u]\!] \cdot [\![v]\!] = \hat{\lambda}_\tau (u,v), \quad \forall\, v \in S_{\underline{p}}(\mathcal{T}) \tag{4}$$

is obtained by restricting the form $\mathcal{B}_\tau$ to $S_{\underline{p}}(\mathcal{T}) \subset V$. This construction provides a framework for the simultaneous abstract analysis of both the continuous as well as the discrete eigenvalue problems. We first establish an estimate of the error in the projection of the companion form from V to $H^1_0(\Omega)$ and call this the nonconformity error and denote it by $\mathcal{R}_{nc}$. We then proceed and estimate the error in the projection of V to $S_{\underline{p}}(\mathcal{T})$ and call this the a posteriori error and denote it by $\mathcal{R}_{ap}(\tau)$. For the a posteriori error component we will also present a computable error estimator using the technique of hierarchical bases. Finally, both estimators are combined in to provide an estimate of the approximation error. In particular,

$$\sum_{i=1}^m \frac{|\hat{\lambda}_{\tau,i} - \lambda_i|}{\lambda_i} \le \frac{C_1}{\tau - 1} \mathcal{R}_{nc} + C_2 \mathcal{R}_{ap}(\tau)$$

is an example of a type of estimates—for the cluster of lowermost eigenvalues $\lambda_1 \le \cdots \le \lambda_m < \lambda_{m+1}$ of (3)—which we will present in Section 4. Further, we will show that by an appropriate choice of τ we may make the a posteriori error the dominant part of the error. For an alternative approach to the analysis of the nonconformity error see [26].

In Section 2 we present main abstract results in the context of discontinuous Galerkin methods and introduce the notation. In Section 3 we review basic facts on discontinuous penalty type Galerkin methods and establish results which guarantee that the concrete formulations satisfy the requirements of the abstract theory. In Section 4 we construct a computable a posteriori error estimator and establish its basic properties. Extensive numerical results will be presented in the subsequent publication.

2. Abstract variational source and eigenvalue problems

Given an open, bounded domain $\Omega \subset \mathbb{R}^d$, let $V = H^1_0(\Omega) + S$, where $S \subset L^2(\Omega)$ is finite dimensional, $S \not\subset H^1_0(\Omega)$ and $H^1_0(\Omega) \cap S \neq \{0\}$. Let $B, J : V \times V \to \mathbb{R}$ be symmetric bilinear forms on V satisfying