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Explicit bound for quadratic Lagrange interpolation constant on triangular finite elements[☆]

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ABSTRACT

For the quadratic Lagrange interpolation function, an algorithm is proposed to provide explicit and verified bound for the interpolation error constant that appears in the interpolation error estimation. The upper bound for the interpolation constant is obtained by solving an eigenvalue problem along with explicit lower bound for its eigenvalues. The lower bound for interpolation constant can be easily obtained by applying the Rayleigh–Ritz method. Numerical computation is performed to demonstrate the sharpness of lower and upper bounds of the interpolation constants over triangles of different shapes. An online computing demo is available at http://www.xfliu.org/onlinelab/.

1. Introduction

In this paper we aim to provide explicit error estimation for quadratic interpolation operator Π_2 defined for functions over triangular elements. Given a triangle T, denote the three vertices by p_1, p_2, p_3, and the mid-points by p_{12}, p_{23}, p_{31}; see Fig. 1.

The finite element method (FEM) defined on a triangulation of domains is often used to solve partial differential equations, for example, the model problem of Poisson's equation. In many cases, one can only make sure the H^2-regularity of the solution u, i.e., $u\in H^2(\Omega)$, and the first and second order Lagrange interpolations Π_1 and Π_2 are often used to give error estimation for FEM solutions; see, e.g., [1,2].

Π_1 interpolation. Let us first review the definition of Π_1: $\Pi_1 u$ is a linear function that interpolates $u\in H^2(T)$ at vertices of T, i.e.,

$$(\Pi_1 u)(p_i) - u(p_i) = 0, \quad i = 1, 2, 3\,. \tag{1}$$

One of the interpolation error estimation for Π_1 is given by

$$|u - \Pi_1 u|_{1,T} \le \widetilde{C}_T |u|_{2,T}, \ \forall u \in H^2(T)\,. \tag{2}$$

The definition of $|\cdot|_i$ $(i=0,1,2,\cdots)$ is taken from the notation of Sobolev spaces; see Section 2. The explicit bound of constant $\widetilde{C}_T$ dates back to the work of Natterer [3] and Lehmann [4], while recent work can be found in Liu and Kikuchi

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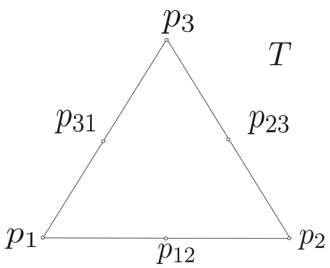

Fig. 1. Configuration of a triangle element T.

Table 1
Verified bound for interpolation constants over different triangles.

(a, b)	λ_{low}	λ_{upper}	C_{low}	C_{upper}
$(0, 1)$	14.8181	15.1101	0.2571	0.2598
$(0, \sqrt{3}/3)$	21.4906	22.1234	0.2125	0.2158
$(1/2, \sqrt{3}/2)$	31.6764	32.2821	0.1759	0.1777
$(-1/2, \sqrt{3}/2)$	5.15806	5.26263	0.4358	0.4404

[5,6] and Kobayashi [7], etc. A sharp bound for the interpolation error constant $\widetilde{C}_T$ on concrete elements is given as follows (see, e.g., [8]),

$$\begin{cases} \widetilde{C}_T \le 0.4889 & \text{for unit isosceles right triangle;} \\ \widetilde{C}_T \le 0.3186 & \text{for unit regular triangle.} \end{cases} \tag{3}$$

Π_2 *interpolation.* The second order Lagrange interpolation Π_2 for $u \in H^2(T)$ is a quadratic polynomial satisfying

$$(\Pi_2 u)(p_i) - u(p_i) = 0, \quad i \in I_0, \tag{4}$$

where I_0 is a set of indices given by $I_0 := \{1, 2, 3, \{12\}, \{23\}, \{31\}\}$. The error estimation for $\Pi_2 u$ is given as follows,

$$|u - \Pi_2 u|_{1,T} \le C_T |u|_{2,T}, \ \forall u \in H^2(T)\,. \tag{5}$$

Here, the interpolation constant C_T only depends on the shape of triangle T itself. For Π_2 and even general kth order Lagrange interpolation, there have been various literatures to investigate the dependence of interpolation error on triangle shapes; see, e.g., the early work of Jamet [9] and recent work of Kobayashi and Tsuchiya [10].

To the author's best knowledge, there is no result reported for the upper bound of the constant C_T. In this paper, we will propose a method that gives explicit bound for C_T by solving eigenvalue problem of corresponding differential operators. As shown in Table 1 of Section 4, we have, for example,

$$\begin{cases} C_T \le 0.2598 & \text{for unit isosceles right triangle;} \\ C_T \le 0.1777 & \text{for unit regular triangle.} \end{cases} \tag{6}$$

Compared with the bound for $\widetilde{C}_T$, one can see that Π_2 has more accurate error estimation than the linear interpolation Π_1.

The method proposed in this paper can also be applied to interpolation for u with higher regularity. For example, for $u \in H^3(T)$, the following error estimation holds,

$$|u - \Pi_2 u|_{i,T} \le C_{T,i} |u|_{3,T}, \ \forall u \in H^3(T) \quad (i = 0, 1, 2)\,. \tag{7}$$

The sharp and explicit values of the interpolation constants $C_{T,i}$ are rarely reported. A brief discussion for bounding such constants can be found in Section 5.

The structure of the rest of the paper is as follows. In Section 2, the characterization of constant C_T through minimization problem and eigenvalue problems is given, where we can see that to give a lower bound of C_T is quite easy. In Section 3, the upper bound of C_T is discussed through the lower bound of certain eigenvalue, which is main contribution of our paper. In Section 4, the computed lower and upper bound for C_T are given for T with various shapes. Finally in Section 5, we provide a rough idea to bound the constants in Eq. (7) and point out the future work.

2. Preliminaries

In this paper, the domain Ω of functions is selected as the triangle T. The standard notation is used for Sobolev function spaces $W^{k,p}(\Omega)$. The associated norms and semi-norms are denoted by $\|\cdot\|_{k,p,\Omega}$, $|\cdot|_{k,p,\Omega}$, respectively (see, e.g., Chapter 1