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On solving the sum-of-ratios problem☆

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ABSTRACT

This paper addresses the development of efficient global search methods for fractional programming problems. Such problems are, in general, nonconvex (with numerous local extremums) and belong to a class of global optimization problems. First, we reduce a rather general fractional programming problem with d.c. functions to solving an equation with a vector parameter that satisfies some nonnegativity assumption. This theorem allows the justified use of the generalized Dinkelbach's approach for solving fractional programming problems with a d.c. goal function. Based on solving of some d.c. minimization problem, we developed a global search algorithm for fractional programming problems, which was tested on a set of low-dimensional test problems taken from the literature as well as on randomly generated problems with up to 200 variables or 200 terms in the sum.

1. Introduction

The fractional optimization is quite attractive and challenging from the view-point of the optimization theory and methods and arises in various economic applications and real-life problems, whenever one or several ratios require optimization. Let us mention a few examples, mainly following the surveys by Schaible [1,2], where numerous other applications can be found. Numerators and denominators in ratios may represent cost, capital, profit, risk or time, etc. Fractional programs are closely related to the associated multiple-objective optimization problem, where a number of ratios are to be maximized simultaneously. Thus, the objective function in a fractional program can be considered as a utility function expressing a compromise between the different objective functions of the multiple-objective problem. Other applications include a multistage stochastic shipping problem [3], profit maximization under fixed cost [4], various models in cluster analysis [5], multi-objective bond portfolio [6], and queuing location problems [7].

It is well-known that without supplementary assumptions the sum-of-ratios program is NP-complete [8]. Surveys on methods for solving this problem can be found in [1,2,9,10]. According to the surveys, the majority of the methods make restrictive assumptions either on the concavity or linearity of the ratios. When the ratios are nonlinear, the most popular techniques are based on the Branch and Bound approach, see e.g. [11,12], and maximization problems with concave-convex ratios are often considered [13]. Therefore, development of new efficient methods for special classes of nonconvex problems, particularly for a fractional program, still remains an important field of research in mathematical optimization [14–17]. Besides, the problems with the functions representable as a difference of convex functions (i.e. d.c. functions) can be considered among the most attractive ones in nonconvex optimization [14–19], because any continuous optimization problem can be approximated by a d.c. problem with any prescribed accuracy [14,16].

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Generalizing the Dinkelbach's idea [20], we propose to address fractional problems with d.c. functions by means of their reduction to solving an equation with the vector parameter under the nonnegativity assumption. This technique is stipulated by the unpredictable behavior and incomputable characters of the ratios in the sum. Moreover, the sum of gradients varies very rapidly. In addition, very often one or several terms of the sum tend to infinity during the computing, thus, it becomes impossible to compute even one term.

Therefore, instead of solving a fractional program directly, we propose to combine the solution of the corresponding d.c. minimization problem with a search with respect to the vector parameter. For this purpose, in Section 2 we introduce this d.c. minimization problem and study the properties of its optimal value. In addition, we prove the existence of the solution to the equation and the reduction theorem. In Section 4 we recall the basic elements of the global search theory for d.c. minimization, which has turned out to be rather efficient in terms of computation [18,19,21–25]. The global search theory comprises the local search methods, the necessary and sufficient global optimality conditions, and the global search algorithm. In Section 5 we introduce the algorithm for finding the vector parameter at which the goal function value of the d.c. minimization problem vanishes. Finally, we have carried out computational experiments that demonstrate that the technique proposed is efficient for solving fractional programs.

2. Statement of the problem

Let us be given an open set $\Omega \in \mathbb{R}^n$ and functions $\phi_i, \psi_i : \Omega \to \mathbb{R}, \; i = 1, \dots, m$, that are continuous on Ω, and a closed set $S \subset \Omega$, such that

$$\phi_i(x) > 0, \;\; \psi_i(x) > 0, \;\; i = 1, \dots, m, \;\; \forall x \in S. \tag{$\mathcal{H}_0$}$$

Note that the case, when $S = \mathbb{R}^n$, is not excluded.

Consider now the following optimization problem [1,9]

$$f(x) := \sum_{i=1}^{m} \frac{\psi_i(x)}{\phi_i(x)} \downarrow \min_x, \;\; x \in S, \tag{$\mathcal{P}$}$$

which further on will be called the basic problem of fractional optimization.

Together with Problem $(\mathcal{P})$, we will also study the parametric optimization problem

$$\Phi(x, \alpha) := \sum_{i=1}^{m} [\psi_i(x) - \alpha_i \phi_i(x)] \downarrow \min_x, \;\; x \in S, \tag{$\mathcal{P}_\alpha$}$$

where $\alpha = (\alpha_1, \dots, \alpha_m)^\top \in \mathbb{R}^m$ is the vector parameter.

Further, let us introduce the function $\mathcal{V}(\alpha)$ of the optimal value to Problem $(\mathcal{P}_\alpha)$ as follows

$$\mathcal{V}(\alpha) := \inf_x \{\Phi(x, \alpha) \mid x \in S\} = \inf_x \left\{ \sum_{i=1}^{m} [\psi_i(x) - \alpha_i \phi_i(x)] \; : \; x \in S \right\}. \tag{1}$$

In addition, suppose that the following assumptions are fulfilled:

$$\left.\begin{array}{l} (a) \;\; \mathcal{V}(\alpha) > -\infty \;\; \forall \alpha \in \mathcal{K}, \text{where } \mathcal{K} \text{ is a convex compact set from } \mathbb{R}^m; \\ (b) \;\; \forall \alpha \in \mathcal{K} \subset \mathbb{R}^m \;\; \text{there exists a solution } z = z(\alpha) \text{ to Problem } (\mathcal{P}_\alpha) \end{array}\right. \qquad \mathcal{H}_1$$

The next result is a generalization of the result from [20]. Let us be given a partial order on the space $\mathbb{R}^m$ in the usual manner [26]. Recall that $\alpha \succeq (\succ) \beta$, if $(\alpha - \beta) \in \mathbb{R}^m_+ (\text{int}\mathbb{R}^m_+)$.

Lemma 1 [27]**.** *The function $\alpha \to \mathcal{V}(\alpha) : \mathbb{R}^m \to \mathbb{R}$ of the optimal value of Problem $(\mathcal{P}_\alpha)$ is*

a) *concave on $\mathbb{R}^m$;*
b) *continuous on $\mathcal{K}$;*
c) *strictly decreasing in the sense of the partial order on $\mathbb{R}^m$ generated by the cone $\mathbb{R}^m_+$, i.e.*

$$\text{if } \alpha \succeq \beta, \;\; \alpha \neq \beta, \;\; \alpha, \beta \in \mathbb{R}^m, \;\; \text{then } \mathcal{V}(\alpha) < \mathcal{V}(\beta). \tag{2}$$

3. Reduction theorem

In this section we present some results concerning the relations between Problems $(\mathcal{P})$ and $(\mathcal{P}_\alpha)$.

Primarily, we give some results on existence of the solution to the equation $\mathcal{V}(\alpha) = 0$.

Proposition 1 [27]**.** *The equation $\mathcal{V}(\gamma) = 0, \; \gamma \in \mathbb{R}^m$, has a unique solution in the sense of the partial order on $\mathbb{R}^m$.*

Proof. Suppose there exist $\alpha, \beta \in \mathbb{R}^m, \; \alpha \succeq \beta, \; \alpha \neq \beta$ satisfying the following equation

$$\mathcal{V}(\alpha) = 0 = \mathcal{V}(\beta). \tag{3}$$

Then, in virtue of Lemma 1 , we obtain $\mathcal{V}(\alpha) < \mathcal{V}(\beta)$, which contradicts (3) in both cases : $0 = \mathcal{V}(\alpha)$ or $\mathcal{V}(\beta) = 0$. □