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Convergence conditions and numerical comparison of global optimization methods based on dimensionality reduction schemes

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ABSTRACT

This paper is devoted to numerical global optimization algorithms applying several ideas to reduce the problem dimension. Two approaches to the dimensionality reduction are considered. The first one is based on the nested optimization scheme that reduces the multidimensional problem to a family of one-dimensional subproblems connected in a recursive way. The second approach as a reduction scheme uses Peano-type space-filling curves mapping multidimensional domains onto one-dimensional intervals. In the frameworks of both the approaches, several univariate algorithms belonging to the characteristical class of optimization techniques are used for carrying out the one-dimensional optimization. Theoretical part of the paper contains a substantiation of global convergence for the considered methods. The efficiency of the compared global search methods is evaluated experimentally on the well-known GKLS test class generator used broadly for testing global optimization algorithms. Results for representative problem sets of different dimensions demonstrate a convincing advantage of the adaptive nested optimization scheme with respect to other tested methods.

1. Introduction

In this paper the black-box global optimization problem

$$f^* = f(y^*) = \min_{y \in P} f(y), \tag{1}$$

$$P = \{y \in \mathbb{R}^N : a_i \le y_i \le b_i,\ 1 \le i \le N\}, \tag{2}$$

is considered as a problem of finding the global minimum value f^* and global minimizers $y^* \in P$ of a real-valued multivariate function $f(y)$ in the hyperparallelepiped (2) of the Euclidean space $\mathbb{R}^N$. The objective function $f(y)$ is supposed to satisfy in the domain P the Lipschitz condition

$$|f(y') - f(y'')| \le L\|y' - y''\|,\ y', y'' \in P, \tag{3}$$

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where $L > 0$ is a finite constant and $\|*\|$ denotes the Euclidean norm in $\mathbb{R}^N$. In general case, these problems are multiextremal and non-smooth.

The global optimization problem under consideration has been drawing attention of many researchers (see, for example, fundamental monographs [1–6]). On the one hand, problems of this kind are very important from the practical point of view because they arise often in scientific and engineering applications (see, for instance, [4,7–13]). On the other hand, Lipschitzian problems (1) and (2) are very interesting for theoretical study because they have a rich variety of properties and, as a consequence, there is no a "universal" algorithm for solving multiextremal problems. These circumstances generate many fruitful approaches for solving this class of problems. Given approaches are based on ideas of different nature (both stochastic [5,6,14–17] and deterministic [18–30]), but, in any case, numerical methods of searching for the global optimum are proposed within the frameworks of approaches as a tool for getting a solution sought.

As it was discussed in [5], the global optimum is an integral characteristic of the problem, i.e., in order to make sure that a point $y^* \in P$ is the global minimizer of the problem (1) and (2) it is required to compare the value $f(y^*)$ with values of the objective function at all points of the domain P, but not in a vicinity of y^* only. As a result, when minimizing an essentially multiextremal function, a numerical global optimization method has to build a grid (random or regular) in the feasible domain and the number of grid nodes increases exponentially when rising the problem dimension. This peculiarity causes the substantial complexity of multiextremal problems and dimension is a crucial factor influencing significantly the efficiency of global optimization algorithms.

In this situation, approaches to elaboration of computational schemes reducing the dimension are widely used in the multiextremal optimization. Here we consider two approaches in which the initial multidimensional problem (1) and (2) is reduced to one or several univariate subproblems solved by efficient one-dimensional algorithms. The first approach applies *the nested optimization scheme* in its classical version [31–37] and in its generalization — *the adaptive nested scheme* [38,39]. The second approach is based on *Peano space-filling curves* mapping the multidimensional domain (2) onto an interval in one-dimensional space $\mathbb{R}^1$ [5,40–46].

The rest of the paper is organized in the following way. Section 2 describes the basic structures of the reduction schemes mentioned above and the univariate characteristical methods of global optimization to be applied within the reduction structures for solving the internal one-dimensional subproblems. Section 3 is devoted to a theoretical substantiation of the convergence for multidimensional methods combining the reduction schemes with characteristical methods of global search and contains both the known and new theoretical results. Section 4 presents experimental results of efficiency comparison for the methods described in previous sections on representative sets of multiextremal test functions of various dimensions belonging to the popular test class GKLS [47] with a controllable complexity. Finally, Section 5 contains a brief conclusion.

2. Schemes of dimensionality reduction

The first approach to dimensionality reduction called the scheme of nested optimization is based on the well-known relation (see, e.g., [5,25,31])

$$\min_{y\in P} f(y) = \min_{y_1\in[a_1,b_1]} \ldots \min_{y_N\in[a_N,b_N]} f(y_1,\ldots,y_N). \tag{4}$$

In order to describe the scheme let us define a family of reduced functions as follows:

$$f^N(y) \equiv f(y), \tag{5}$$

$$f^i(y_1,\ldots,y_i) = \min_{y_{i+1}\in[a_{i+1},b_{i+1}]} f^{i+1}(y_1,\ldots,y_i,y_{i+1}),\ 1 \le i \le N-1. \tag{6}$$

Then, according to (4), in order to find the solution to the multidimensional problem (1) and (2) it is sufficient to solve the one-dimensional problem

$$f(y^*) = \min_{y_1\in[a_1,b_1]} f^1(y_1). \tag{7}$$

But in order to evaluate the function f^1 at a fixed point y_1 it is necessary to solve the one-dimensional problem of the second level

$$f^1(y_1) = \min_{y_2\in[a_2,b_2]} f^2(y_1,y_2), \tag{8}$$

and so on up to the univariate minimization at the Nth level the function $f^N(y) = f(y)$ with fixed coordinates $y_1,\ldots,y_{N-1}$.

Thus, instead of the multidimensional problem (1) and (2) one can solve the set of nested one-dimensional subproblems

$$\min_{y_i\in[a_i,b_i]} f^i(y_1,\ldots,y_{i-1},y_i),\ 1 \le i \le N, \tag{9}$$

in which the coordinates $y_1,\ldots,y_{i-1}$ have been already determined by the subproblems of preceding levels. The objective functions $f^i(y_1,\ldots,y_i)$ of the subproblems (9) satisfy the Lipschitz condition for the corresponding argument y_i subject to the assumption (3) for the function $f(y)$ (see [48]). Therefore, efficient univariate optimization algorithms can be taken for