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Nonlinear programming and Grossone: Quadratic Programing and the role of Constraint Qualifications

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ABSTRACT

A novel and interesting approach to infinite and infinitesimal numbers was recently proposed in a series of papers and a book by Sergeyev. This novel numeral system is based on the use of a new infinite unit of measure (the number *grossone*, indicated by the numeral ①), the number of elements of the set, ℕ, of natural numbers. Based on the use of ①, De Cosmis and De Leone (2012) have then proposed a new exact differentiable penalty function for constrained optimization problems. In this paper these results are specialized to the important case of quadratic problems with linear constraints. Moreover, the crucial role of Constraint Qualification conditions (well know in constraint minimization literature) is also discussed with reference to the new proposed penalty function.

1. Introduction

In a series of papers and in a book [2–5], Sergeyev proposed an interesting and fresh approach to infinite and infinitesimal numbers whose peculiar characteristic is the attention to numerical aspects and to applications. This novel numeral system is based on the use of a new infinite unit of measure (the numeral *grossone*, indicated by ①), the number of elements of the set, ℕ, of natural numbers. Grossone is introduced through the following three properties:

- *Infinity.* Any finite natural number n is less than grossone, i.e., $n <$ ①.
- *Identity.* The following relationships link ① to the identity elements 0 end 1

$$0 \cdot ① = ① \cdot 0 = 0,\ ① - ① = 0,\ \frac{①}{①} = 1,\ ①^0 = 1,\ 1^{①} = 1, \quad 0^{①} = 0 \tag{1}$$

- *Divisibility.* For any finite natural number n, the sets $\mathbb{N}_{k,n}$, $1 \le k \le n$,

$$\mathbb{N}_{k,n} = k, k+n, k+2n, k+3n, \ldots, \quad 1 \le k \le n, \quad \bigcup_{k=1}^{n} \mathbb{N}_{k,n} = \mathbb{N} \tag{2}$$

have the same number of elements indicated by $\frac{①}{n}$.

It should be noted that ① is not a symbol and is not used to make symbolic calculation. In fact, the new numeral ① belongs to ℕ, and it has both cardinal and ordinal properties, exactly as the "standard", finite natural numbers. The approach proposed by Sergeyev (see in particular [5]) and employed here is more computationally and practically oriented. The same approach used in chemistry or physics and based on a strict correlation between researcher, object of investigation and tools

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utilized to investigate the object or the phenomenon, is proposed also in mathematics. In this context, a new positional numeral system with base ① is introduced, where the number

$$C = c_{p_m}①^{p_m} + c_{p_{m-1}}①^{p_{m-1}} + \cdots + c_{p_1}①^{p_1} + c_{p_0}①^{p_0} + c_{p_{-1}}①^{p_{-1}} + \cdots + c_{p_{-k}}①^{p_{-k}} \quad (3)$$

is represented by the record

$$c_{p_m}①^{p_m} \ldots c_{p_1}①^{p_1} c_{p_0}①^{p_0} c_{p_{-1}}①^{p_{-1}} \ldots c_{p_{-k}}①^{p_{-k}} \quad (4)$$

where

$$p_m > p_{m-1} > \cdots > p_1 > p_0 = 0 > p_{-1} > \cdots > p_{-k}.$$

This numeral system for expressing numbers are the tools for observation, and, by using a more powerful numerical system, it is possible to reach more precise results in applied and pure mathematics. The numerals $c_i \neq 0$, belonging to the "traditional" numeral system are called *grossdigits*. Numbers p_i are called *grosspowers* and can be finite, infinite, and infinitesimal. The term having $p_0 = 0$ represents the finite part of C, the terms having finite positive grosspowers represent the infinite part of C, and terms having negative finite grosspowers represent the infinitesimal parts of C. The term $①^{-1}$ is an infinitesimal.

The new methodology is under study from both theoretical and applied viewpoints. On the one hand, many authors connected the new approach to the historical panorama of ideas dealing with infinities and infinitesimals [6–11]. In particular, relations of the new approach to bijections are studied in [8] and metamathematical investigations on the new theory and its non-contradictory can be found in [7]. On the other hand, the new methodology has been successfully applied in many areas such as cellular automata [12–14], Euclidean and hyperbolic geometry [15,16], percolation [17–19], fractals [19–23], numerical differentiation and optimization [24,25], infinite series and the Riemann zeta function [26–28], the first Hilbert problem, Turing machines, and lexicographic ordering [11,29–32], ordinary differential equations [33–36], etc. The interested reader is invited to have refer to the surveys [37,38] and to the book [2] for an introduction for general public.

Based on this new numeral system, various applications to linear and nonlinear problems have been proposed in [1]. In particular, the relationships of KKT points [39] of the constrained optimization problem

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{subject to} & g(x) \leq 0 \\ & h(x) = 0 \end{array} \quad (5)$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ and stationary points of the unconstrained problem

$$\min_x f(x) + \frac{1}{2}①\|\max\{0, g(x)\}\|^2 + \frac{1}{2}①\|h(x)\|^2 =: F(x) \quad (6)$$

have been investigated.

In this paper we discuss two different aspects of the use of ① in constrained minimization problems.

Firstly in Section 2 we discuss the importance of Constraint Qualification (whose role is well known and studied in nonlinear optimization literature) when ① is utilized to move from constrained to unconstrained problems. It is well known that exactness in (6) cannot be achieved if we use, instead of ①, any finite real value. Moreover, differentiable exact penalty functions can be constructed [40], but it is necessary to include terms related to first order optimality conditions thus making the penalty function much more complicate. The use of ① provides a very simple, but powerful alternative. However, Constraint Qualification play a fundamental role even in this context. An example shows that, when Constraint Qualifications do not hold, spurious or incorrect solutions can arise from stationary point of the unconstrained problem.

In Section 3, following De Cosmis and De Leone [1], again we use ① to define an unconstrained minimization problem for a quadratic problem with linear constraints. Here, Constraint Qualification conditions are trivially satisfied, since all constraints are linear. We will show that the finite term of a stationary point for the unconstrained problem provides a solution of the constrained problem, while the multipliers can be easily and directly obtained from the $①^{-1}$ terms.

We briefly describe our notation now. All vectors are column vectors and will be indicated with lower case Latin letter $(x, z, \ldots)$. Subscripts indicate components of a vector, while superscripts are used to identify different vectors. Matrices will be indicated with upper case roman letter $(A, B, \ldots)$. The set of real numbers and the set of nonnegative real numbers will be denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. The rank of a matrix A will be indicated by rank A. The space of the n-dimensional vectors with real components will be indicated by $\mathbb{R}^n$ and $\mathbb{R}^n_+$ is an abbreviation for the nonnegative orthant in $\mathbb{R}^n$. The symbol $\|x\|$ indicates the Euclidean norm of a vector x. Superscript ^{T} indicates transpose. The scalar product of two vectors x and y in $\mathbb{R}^n$ will be denoted by x^Ty. Here and throughout the symbols $:=$ and $=:$ denote definition of the term on the left and the right sides of each symbol, respectively. The gradient $\nabla f(x)$ of a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ is assumed to be a column vector.

2. The importance of Constraint Qualifications

It is well know that the Karush–Kuhn–Tucker (KKT) first order optimality conditions for constrained optimization strongly depend on specific conditions on the constraints of the problem, known as Constraint Qualifications (CQs).