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Lexicographic multi-objective linear programming using grossone methodology: Theory and algorithm

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ABSTRACT

Numerous problems arising in engineering applications can have several objectives to be satisfied. An important class of problems of this kind is lexicographic multi-objective problems where the first objective is incomparably more important than the second one which, in its turn, is incomparably more important than the third one, etc. In this paper, Lexicographic Multi-Objective Linear Programming (LMOLP) problems are considered. To tackle them, traditional approaches either require solution of a series of linear programming problems or apply a scalarization of weighted multiple objectives into a single-objective function. The latter approach requires finding a set of weights that guarantees the equivalence of the original problem and the single-objective one and the search of correct weights can be very time consuming. In this work a new approach for solving LMOLP problems using a recently introduced computational methodology allowing one to work numerically with infinities and infinitesimals is proposed. It is shown that a smart application of infinitesimal weights allows one to construct a single-objective problem avoiding the necessity to determine finite weights. The equivalence between the original multi-objective problem and the new single-objective one is proved. A simplex-based algorithm working with finite and infinitesimal numbers is proposed, implemented, and discussed. Results of some numerical experiments are provided.

1. Introduction

Engineering applications often lead to optimization problems where several objectives should be satisfied. An important class of problems of this kind is lexicographic multi-objective problems where the first objective is incomparably more important than the second one which, in its turn, is incomparably more important than the third one, etc. In case each of the objectives is represented by a linear function under linear constraints, Lexicographic Multi-Objective Linear Programming (LMOLP) problems are considered. Traditionally LMOLP problems are solved in two different ways (see, e.g., [7,14,36,37]). The first one called *preemptive approach* consists of solving a sequence of single-objective linear programming (LP) problems,

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where new constraints are added to the subsequent LP problem once an optimal solution to the previous problem has been found. Clearly, this approach is time consuming. The second approach known as *nonpreemptive* one (see [36]) transforms LMOLP into a single-objective LP problem by using a weighted sum of the objectives. This approach has the difficulty to find the weights which guarantee, a-priori, the equivalence of the resulting single-objective problem with the original multi-objective one. In practice, determining such weights is a tricky task and it can be even more time consuming then the preemptive approach.

In the present paper, it is proposed to analyze LMOLP problems using a recently introduced computational methodology allowing one to work *numerically* with infinities and infinitesimals in a handy way (see for a detailed introduction surveys [19,23,28,30,32] and the book [17] written in a popular way). This computational methodology has already been successfully applied in optimization and numerical differentiation (see [3,6,24,40]) and in a number of other theoretical and computational research areas such as cellular automata (see [4,5]), Euclidean and hyperbolic geometry (see [11,12]), percolation (see [8,9,38]), fractals (see [18,20,26,31,38]), infinite series and the Riemann zeta function (see [21,25,39]), the first Hilbert problem, Turing machines, and supertasks (see [15,22,33,34]), numerical solution of ordinary differential equations (see [1,13,27,35]), etc.

This methodology uses a numeral system working with an infinite number called *grossone*, expressed by the numeral ①, and introduced as the number of elements of the set of natural numbers (the non-contradictory of the methodology has been studied in [10]). This numeral system allows one to express a variety of numbers involving different infinite and infinitesimal parts and to execute operations with all of them in a unique framework. Notice that this numerical approach is not related to the famous non-standard analysis (see [16]) that has a symbolic character and, therefore, allows symbolic computations only, whereas the present text is dedicated to numerical optimization methods.

Following the guidelines traced in [2,29] for working with problems involving lexicographic ordering, it is proposed hereinafter to transform LMOLP into a single-objective LP problem by multiplying the most important objective by 1, the second by $①^{-1}$, the third by $①^{-2}$ etc., where $①^{-i}, 1 < i \le r$, are infinitesimals and r is a finite number of objectives. It is shown then that, after this transformation, the resulting single-objective LP problem formulated with the help of *grossone*-based numbers can be solved only once by using a simplex-like method working with *grossone*-based numbers that can include infinitesimal parts. The overall advantage of this approach consists of the possibility to solve only one LP problem, without the need to look for correct finite weights that should be provided to an algorithm if the traditional nonpreemptive scheme is applied.

The remaining text is structured as follows. In Section 2, the lexicographic multi-objective linear programming problem is stated and the preemptive and nonpreemptive schemes are described. The *grossone* methodology is briefly presented in Section 3 whereas Section 4 introduces the nonpreemptive *grossone*-based scheme. In Section 5, a theoretical analysis of the introduced approach is performed and the equivalence between the original multi-objective problem and the new single-objective one is proved. Section 6 provides the *gross*-simplex algorithm able to solve the resulting problem. Finally, Section 7 presents some promising experimental results.

2. Lexicographic multi-objective linear programming

Given the LMOLP problem:

$$\begin{aligned} &\text{LexMax } \mathbf{c}^1 \cdot \mathbf{x},\ \mathbf{c}^2 \cdot \mathbf{x}, \ldots, \mathbf{c}^r \cdot \mathbf{x} \\ &\quad \text{s.t. } \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqslant 0\} \end{aligned} \tag{P1}$$

where $\mathbf{c}^i$, $i = 1, \ldots, r$, are row vectors $\in \mathbb{R}^n$, $\mathbf{x}$ is a column vector $\in \mathbb{R}^n$, $\mathbf{A}$ is a full-rank matrix $\in \mathbb{R}^{m \times n}$, $\mathbf{b}$ is a column vector $\in \mathbb{R}^m$, and $\cdot$ is the standard scalar product between two real vectors. LexMax in (P1) denotes *Lexicographic Maximum* and means that the first objective is much more important than the second, which is, on its turn, much more important than the third one, and so on. Sometimes in literature this is denoted as $\mathbf{c}^1 \cdot \mathbf{x} \gg \mathbf{c}^2 \cdot \mathbf{x} \gg \cdots \gg \mathbf{c}^r \cdot \mathbf{x}$. As in any LP problem, the domain of (P1) is a polytope:

$$S \equiv \{x \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqslant 0\}. \tag{1}$$

Notice that the formulation of (P1) makes no use of *gross*-numbers or *gross*-arrays involving ①, namely, it involves finite numbers only. Hereinafter we assume that S is bounded and non-empty. In the literature (see, e.g., [7,14,36,37]), there exists two approaches for solving the problem (P1): the *preemptive scheme* and the *nonpreemptive scheme*. They are described in the following two subsections.

2.1. The preemptive scheme

The preemptive scheme introduced in [36] is an iterative method that attacks (P1) by solving a series of single-objective LP problems. It starts by considering the first objective function alone, i.e., by solving the following problem:

$$\begin{aligned} &\text{Max} \quad \mathbf{c}^1 \cdot \mathbf{x} \\ &\text{s.t.} \quad \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqslant 0\} \end{aligned} \tag{P2.1}$$