Applied Mathematics and Computation 000 (2017) 1–8

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

The Sierpinski curve viewed by numerical computations with infinities and infinitesimals

Fabio Caldarola

Department of Mathematics and Computer Science, Cubo 31/B, University of Calabria, Rende (CS) 87036, Italy

A R T I C L E I N F O

Article history:
Available online xxx

Keywords:
Sierpinski curve
Space-filling curves
Fractals
Fractal geometry
Numerical infinities and infinitesimals
Grossone

A B S T R A C T

The Sierpinski curve is one of the most known space-filling curves and one with the highest number of applications. We present a recently proposed computational methodology based on the infinite quantity called *grossone* to investigate the behavior of two different constructions of the Sierpinski curve. We emphasize that, adopting this point of view, we have infinitely many Sierpinski curves depending, contrarily to traditional analysis, on each specific starting configuration. Of particular interest are some power series expansions in the new infinitesimal quantities emerging from the study of the considered curves.

1. Introduction

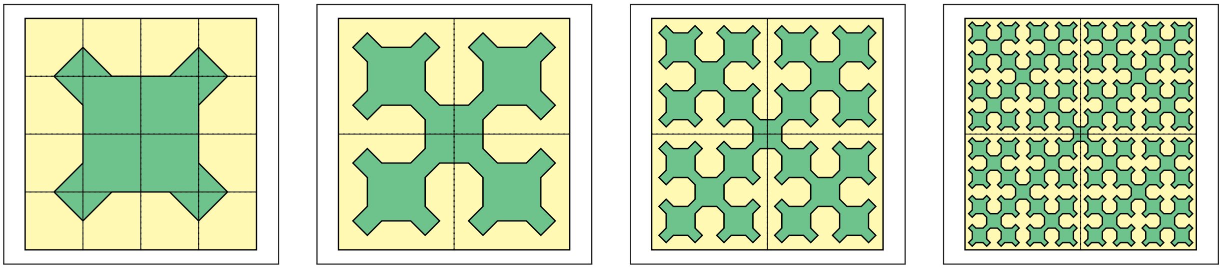
The *Sierpinski curve*, also called the *Sierpinski's square snowflake*, discovered by the Polish mathematician Waclaw Sierpinski in 1912 ([39–41]), is one of the most known space-filling curves (see, for instance the cover of [2], [17], and many other books on the topic) and probably the most important for applications in Physics, Engineering, Optimization Theory, Computer Science, Signal Theory, etc., because of its great symmetry.

There are several possible constructions of the Sierpinski curve but in this article we consider only two of them, certainly the most known, which are so similar each other than are often confused: a secondary purpose of this paper is to make clearer, by a direct comparison, their constructive process together with their computational characteristics, emphasizing the differences produced by the two methods, both geometrically and analytically. The aim of our work is instead a deep study and a detailed investigation on the behavior "at infinity" of two such constructions of the Sierpinski curve, that is, when the building process reaches an infinite number of steps. From this point of view, our work is very innovative and also quite unusual; in fact, by using only traditional analysis, a description of such a behavior at infinity reduces to compute only four trivial limits in two lines of space, as shown in (6) and (20). (Of course, by "traditional analysis" we mean the one as developed from Weierstrass on.) It is evident that, in this way, we lose almost all information because traditional analysis does not have an adequate language and appropriate notations to describe in detail the phenomena, as occurring outside the narrow circle of finite numerical values. We will show as, adopting a new computational system which allows to distinguish a whole family of infinitely many different infinitesimal and infinite values and to do computations with them as with real numbers, we can describe a very rich behavior, full of meaning, for the Sierpinski curve at infinity, as and maybe more, than in its finite building steps.

The new computational system we referred before was introduced recently by Y.D. Sergeyev; for some detailed introduction surveys which show how to work *numerically* with infinities and infinitesimals in such a handy way, the reader can consult [22,27,32,33] and also the book [18] written in a popular way. This new computational methodology has already

E-mail address: caldarola@mat.unical.it

http://dx.doi.org/10.1016/j.amc.2017.06.024
0096-3003/



(a) S_0 (b) S_1 (c) S_2 (d) S_3

Fig. 1. The first four steps in the construction of the Sierpinski curve starting from a square of side a.

been successfully applied in a number of theoretical and computational research areas as optimization theory and numerical differentiation (see [5,9,10,28,44]), Euclidean and hyperbolic geometry (see [13,14]), cellular automata (see [7,8] and in the context of [3] under investigation), percolation (see [11,12,42]), fractals (see [4,42,21,23,30,34]), conditionally convergent numerical series and the Riemann zeta function (see [24,29,43]), Hilbert problems, Turing machines and supertasks (see [16,26,35–37]), numerical solution of ordinary differential equations (see [1,15,31,38]), etc.

In particular, in [34] the author uses this new numerical system to compute the exact perimeter and the area of the Koch snowflake and he points out that, unlike classical analysis, he finds infinitely many Koch snowflakes depending on the starting configuration. This is a further example of the new potentialities arising from the employ of more appropriate and powerful tools, and we think that it is also even more interesting using the new methodological approach to study curves that fill a two or a three dimensional space. In this regard we will discover, among other things, as in both the constructions considered here, there is a great sensitive dependence of the system from the starting configuration, and we will show as the remarkable precision and the accuracy achieved in dealing with infinite numbers, opens the way for a whole new range of possible researches and investigations. For instance, being called *grossone* the main basic infinite entity of Sergeyev's new system, the filling curves considered in this paper leads us to produce various power series expansions whose variables are *grossone-based infinitesimal quantity*, and this is an absolute novelty which is unprecedented even in the new theory. These types of grossone-based power series look very attractive and interesting in several possible other contexts, and they seem to suggest new lines of fruitful researches about series involving such infinitesimals and infinities.

2. The Sierpinski curve: the first construction

As we said in the Introduction, there are several constructions for the Sierpinski curve and the final result differs slightly depending on the used procedure. Now we go to explain the first constructive method for which we refer to [6]: we begin from a square with side of length a, then we divide it into sixteen smaller squares q_0 of side $d_0 = a/4$ and we join up the midpoints of some edges as in Fig. 1(a) obtaining the zero-step curve S_0. To get S_1 we consider the original square divided into four parts, in each of these corners we construct a curve like S_0 on half the scale and we unite them in the center as shown in Fig. 1(b). Then the process is repeated in the same way, to construct the curves S_2, S_3, etc., as shown in Fig. 1(c) and (d). Finally the Sierpinski's curve S is defined as the limit $\lim_{n\to\infty} S_n$.

We highlight that to obtain the curve S_n we have to connect four curves like S_{n-1} but on half scale using four segments of length $d_n = \frac{a}{4} \cdot \frac{1}{2^n}$ $(n \geqslant 0)$, so if l_n and A_n are the length and the area, respectively, of the closed curve S_n, we find, by an easy computation, the following recursive formulas

$$\begin{cases} l_0 = 4d_0 + 12 \cdot \dfrac{d_0}{2}\sqrt{2} = \dfrac{a}{2}\left(2 + 3\sqrt{2}\right) \\ l_n = 4 \cdot \dfrac{1}{2}\, l_{n-1} - \dfrac{d_n}{2}\sqrt{2} \cdot 4 + 4d_n = 2l_{n-1} + \dfrac{a}{2^{n+1}}\left(2 - \sqrt{2}\right), \quad \forall n \geqslant 1 \end{cases} \tag{1}$$

$$\begin{cases} A_0 = (2d_0)^2 + 12 \cdot \dfrac{1}{4}\left(\dfrac{d_0}{2}\sqrt{2}\right)^2 = \dfrac{11}{32}a^2 \\ A_n = 4 \cdot \dfrac{a_{n-1}}{4} + (2\,d_n)^2 - \left(\dfrac{d_n}{2}\right)^2 \cdot 2 = A_{n-1} + \dfrac{7}{32}a^2 \cdot \dfrac{1}{4^n}, \quad \forall n \geqslant 1. \end{cases} \tag{2}$$

From (1) then we obtain

$$l_n = \frac{2^{n+2}a}{3}\left(1 + \sqrt{2}\right) - \frac{a}{2^{n+1} \cdot 3}\left(2 - \sqrt{2}\right), \tag{3}$$