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Acoustic metamaterial models on the (2+1)D Schwarzschild plane

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ABSTRACT

Recent developments in acoustic metamaterial engineering have led to the design and fabrication of devices with formidable properties, such as acoustic cloaking, superlenses and ultra-sound waves. Artificial materials of this type are generally absent in natural environments. In this work, we focus on feasible implementations of acoustic black holes on the 2D plane, that is, within (2+1)D spacetime. For an accurate description of planar black holes in transformation acoustics, we examine Schwarzschild-type models. After proposing an appropriate form for the Lorentzian metric of the underlying spacetime, we explore the geometric content and physical consequences of such models, which will turn out to have de Sitter and anti-de Sitter spacetime structure. For this purpose, we derive a general expression for its acoustic wave propagation. Next, a numerical simulation is carried out for prototype waves which probe these spacetime geometries. Finally, we discuss how to fine-tune the corresponding acoustic parameters for an implementation in the laboratory environment.

1. Introduction

Metamaterials are artificially manufactured materials which by far surpass the properties of conventional materials found in nature. With their help researchers and engineers alike are presented with unique possibilities for the development of novel artificial devices with extraordinary characteristics. This does not merely entail a simple and gradual improvement of devices with already known features, but involves a paradigm shift, *e.g.* with optical metamaterials it has become possible to construct devices with negative refractive index [1]—a concept which was traditionally regarded as impossible, although already hypothesized in the late 1960s [2].

For almost two decades researchers now have focussed on optical metamaterials, whereas the manipulation of sound waves via acoustic metamaterials only recently has come under their spotlight [3–6]. Acoustic metamaterials allow to model sophisticated acoustic phenomena via curved background spacetimes and make predictions for future laboratory experiments. Apart from the interesting technical applications, these models may also help to settle fundamental questions with far-reaching impact, as *e.g.* the existence and analysis of inverse Doppler effects [7] and Hawking radiation from acoustic black holes [8].

With regard to black holes, one of the first—and remarkably simple—solutions of Einstein's equations for the curved spacetime of the gravitational field with underlying static and spherical symmetry is the Schwarzschild solution [9]. In the

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beginning considered to be a mathematical curiosity and only of academic interest, it has now in the age of high-precision GPS navigation and black-hole astronomy become the centre stage of many practical and important applications.

In order to implement 2D artificial black holes for acoustic waves, several experimental and theoretical pathways have been studied and proposed in the literature [10–13]. In this context, acoustic black holes have been loosely defined as metadevices where sound is effectively trapped in a stream-like event horizon, however, without further precise assumption about the global spacetime structure. In the present work, we employ a different approach to model acoustic wave propagation on a curved spacetime with the characteristics of a genuine Schwarzschild spacetime structure. This approach is based on a variational principle in combination with the powerful framework of differential geometry [14–17].

The Schwarzschild geometry is both a mathematically and physically intriguing non-euclidean geometry and as such a fascinating candidate for the implementation and study of an acoustic metamaterial. In general, for the $(n+1)$D case, it represents a modification of flat Minkowski spacetime which imposes full spherical symmetry for the n-dimensional spatial part.

First, we will briefly review the field formulation of acoustics and its variational principle. Then, before beginning the discussion on the modelling of acoustic wave propagation on the Schwarzschild plane, we examine the feasibility of Schwarzschild-type geometries in (2+1) spacetime dimensions and give a classification of the possible solutions. Next, we outline how to derive within this framework the partial differential equation for the acoustic potential which simulates wave propagation on the Schwarzschild plane. A numerical simulation for prototype waves probing these spacetime and an analysis of their crucial geometric features follows. Finally, we will comment on the design and implementation of such a spacetime with acoustic metadevices. Employing the constitutive equations [15] will enable us to connect the Schwarzschild geometry with the acoustic parameters of the model.

2. Field formulation of acoustics and variational principle

This outline on the field formulation of acoustics and its variational principle closely follows Refs. [15,17]. The importance of variational principles in classical and field mechanics, including optics and electrodynamics, lies in defining concisely and in a coordinate-independent manner the fundamental laws which they describe via a given scalar Lagrange function $\mathscr{L}$, *i.e.* they remain invariant with respect to arbitrary transformations of the coordinates. The equations of motion that fully determine the physical behaviour of the system correspond to the extremal solutions of the action integral $\mathscr{A}$ defined by $\mathscr{L}$. This Lagrangian approach allows to easily reveal the underlying symmetries and conservation laws of the theoretical model via Noether's theorem (see *e.g.* [18]). Furthermore, the laws describing linear physical phenomena will have their equivalent in equations of motion with self-adjoint differential operators acting on the related field variables [19, p. 301]. In principle, this yields separable partial differential equations which are Sturm–Liouville problems for one of the field variables with analytical or at least semi-analytical solutions.

Let acoustics be described by the acoustic potential $\phi : M \to \mathbb{R}$, where M is a smooth spacetime endowed with a Lorentzian metric $\boldsymbol{g}$ having negative signature, *i.e.* $g = \det \boldsymbol{g} < 0$. Then, we postulate that the following action integral is stationary with respect to variations of the potential [15]:

$$\frac{\delta}{\delta\phi}\mathscr{A}[\phi] = \frac{\delta}{\delta\phi}\int_{\Omega} \mathrm{dvol}_g \mathscr{L}(x, \phi, \nabla\phi) = 0 \quad \text{so that} \quad \frac{\delta}{\delta\phi}\mathscr{A}[\phi] = 0. \tag{1}$$

The integration domain $\Omega \subseteq M$ is an open, connected subset of spacetime with smooth boundary $\partial\Omega$, and the corresponding invariant volume element is denoted by $d\mathrm{vol}_g = \sqrt{-g}\, dx^0 \wedge \cdots \wedge dx^3$, where $x \in M$. Here, in general, the Lagrangian is a function $\mathscr{L} : TM \to \mathbb{R}$, where TM is the tangent bundle of coordinate space M. The form of $\mathscr{L}$ is severely constrained by fundamental symmetry requirements: energy–momentum conservation, locality, and free-wave propagation. Its simplest possible choice is [15]

$$\mathscr{L}(\nabla\phi) = \frac{1}{2}\boldsymbol{g}(\nabla\phi, \nabla\phi) = \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}. \tag{2}$$

Note that if $\boldsymbol{v}$ denotes the local fluid velocity, p the acoustic pressure, ϱ_0 the density, and $c > 0$ the time-independent wave speed of the acoustic metamaterial, the gradient or covariant derivative appearing in Eq. (2) will be

$$\phi_{,\mu} = \begin{pmatrix} p/(c\varrho_0) \\ -\boldsymbol{v} \end{pmatrix}. \tag{3}$$

Here, Greek tensor indices refer to the full range of spacetime components, whereas Latin indices will only indicate spatial components. We also adopt the standard notation using a comma and semicolon before the indices to denote partial and covariant derivatives, respectively. In this notation, for a scalar ϕ, the (covariant) components of the derivative $\nabla\phi$ are identically $\phi_{;\mu}$ or $\phi_{,\mu}$.

Expression Eq. (3) encapsulates elementary relations of acoustics [20] and is valid within a fixed laboratory frame. Note that in classical acoustics the four-vector $\phi_{,\mu}$ obviously cannot be fully relativistic, meaning that one cannot transform from one arbitrary inertial laboratory frame to another. However, the 4-vector will transform with a subgroup of the full Lorentz group, according to $\mathrm{SO}^+(1,3)/\mathrm{H}^3$, which excludes all boosts (corresponding to rotations in real hyperbolic 3-space H^3) from the restricted Lorentz group.