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Simplified reproducing kernel method and convergence order for linear Volterra integral equations with variable coefficients

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ABSTRACT

This paper proposes a simplified reproducing kernel method to solve the linear Volterra integral equations with variable coefficients. The main idea of the method is to establish a reproducing kernel direct space that can be used in Volterra integral equations. And in the first time, this paper analyzes the convergence order and stability of the approximate solution. Then the uniform convergence of the numerical solution is proved, and the time consuming Schmidt orthogonalization process is avoided. The proposed method is proved to be stable and is not less than the second order convergence. The algorithm is proved to be feasible and stable through some numerical examples.

1. Introduction

The integral equations is an important branch of modern mathematics, many mathematical and physical problems need to be solved by integral equations or differential equations. The type of integral equations depending on the structure of integrals, for example, Fredholm integral equations, Volterra integral equations and Fredholm–Volterra integral equations. The model of Fredholm and Volterra integro-differential equations extends to every field of application, such as wind ripple in the desert, nano-hydrodynamics and drop wise condensation [1–5]. However, it is usually difficult to get an analytic solution of the integral and integro-differential equations, therefore, many researchers have extensively studied the numerical methods of Volterra integral equations in recent years [6–10]. F. Mirzaee [11] used the rationalized Haar functions to solve the system of linear Volterra integral equations. L.H. Yang [12,13] provide a reproducing kernel method for solving the system of the Volterra integral equations. F. Mirzaee [14] solved the systems of linear Volterra integral equations based on the Euler matrix method. A collocation approach have been used for solving linear Volterra integral equations [15]. An expansion method is used for treatment of second kind Volterra integral equations system [16]. E. Hesameddini [17] solved the Volterra–Fredholm integral equations based on Bernstein polynomials and hybrid Bernstein Block-Pulse functions. F. Mirzaee [18] contributes an efficient numerical approach to solve the systems of high-order linear Volterra integro-differential equations with variable coefficients under the mixed conditions.

In this paper, by simplified reproducing kernel method, we get an approximate solution for linear Volterra integral equations with variable coefficients as follows:

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$$\begin{cases} a_{11}(x)f_1(x) - b_{11}\displaystyle\int_0^x k_{11}(x,t)f_1(t)dt + a_{12}(x)f_2(x) - b_{12}\displaystyle\int_0^x k_{12}(x,t)f_2(t)dt = u_1(x) \\ a_{21}(x)f_1(x) - b_{21}\displaystyle\int_0^x k_{21}(x,t)f_1(t)dt + a_{22}(x)f_2(x) - b_{22}\displaystyle\int_0^x k_{22}(x,t)f_2(t)dt = u_2(x) \end{cases} \tag{1}$$

where $a_{ij}(x)$, $i, j = 1, 2$ are arbitrary smooth functions defined on the interval [0,1], b_{ij}, $i, j = 1, 2$ are given constants.

As known to all, the application of reproducing kernel method for integral and differential equations has been developed by many researchers because this method is easy to obtain the exact solution with the series form and to get approximate solution with higher precision [19–22]. Moreover, more and more scholars use the reproducing kernel method to solve the problem of integral–differential equations [12,13]. The traditional reproducing kernel method is very complicated because it contains Schmidt orthogonalization process. The simplified regenerative kernel method proposed in this paper avoids Schmidt's orthogonalization process and eliminates need to calculate individual reproducing kernel functions, which makes it more widely applicable.

The aim of this paper is to derive the numerical solutions of Eqs. (1) in Section 1. In Section 2, we introduce the reproducing kernel direct space for solving problems. Some primary results are analyzed in Section 3. The numerical algorithm of approximate solution is presented in Section 4. Section 5 describes the convergence order and stability analysis of approximate solution. In Section 6, the presented algorithms are applied to some numerical experiments. Then we end with some conclusions in Section 7.

2. The reproducing kernel direct space

In this section, the reproducing kernel space is given, and the reproducing kernel direct space is defined that we need. We assume that Eqs. (1) have the unique solution.

- Reproducing kernel space $W_2[0, 1]$ is defined as

$W_2[0, 1] = \{u(x)|u'$ is an absolutely continuous real value function,$u'' \in L^2[0, 1]\}$ [20]. The inner product and norm are given by Ref. [20].

- Reproducing kernel space $W_1[0, 1]$ is defined as

$W_1[0, 1] = \{u(x)|u$ is an absolutely continuous real value function,$u' \in L^2[0, 1]\}$ [20]. The inner product and norm are given by Ref. [20].

The reproducing kernel spaces are W_2 and W_1 with reproducing kernel $R_t(x)$ and $r_t(x)$, respectively.

In this paper, consider that the exact solution of Eqs. (1) is a function vector, so, we structure a reproducing kernel direct space, introduce product and norm.

Definition 2.1. The linear space $W_{(2,2)}$ is defined as

$$W_{(2,2)}[0, 1] = W_2[0, 1] \oplus W_2[0, 1] = \{\mathbf{F}(x) = (f_1(x), f_2(x))^T | f_1(x), f_2(x) \in W_2[0, 1]\}.$$

The inner product and norm are defined by

$$\langle \mathbf{F}(x), \mathbf{G}(x)\rangle_{W_{(2,2)}} = \langle f_1(x), g_1(x)\rangle_{W_2} + \langle f_2(x), g_2(x)\rangle_{W_2} \tag{2}$$

$$\|\mathbf{F}(x)\|^2_{W_{(2,2)}} = \|f_1(x)\|^2_{W_2} + \|f_2(x)\|^2_{W_2}$$

Theorem 2.1. *The space $W_{(2,2)}[0, 1]$ is a Hilbert space.*

Proof. Suppose that $\{\mathbf{F}_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $W_{(2,2)}[0, 1]$, however,

$$\mathbf{F}_n(x) = (f_{1,n}(x), f_{2,n}(x))^T, n = 1, 2, \ldots$$

so, $\{f_{1,n}(x)\}_{n=1}^{\infty}$ and $\{f_{2,n}(x)\}_{n=1}^{\infty}$ are Cauchy sequences in W_2, respectively.

Notice that W_2 is a reproducing kernel space, so, there are two functions $g_1(x), g_2(x) \in W_2$, make

$$\|f_{1,n}(x) - g_1(x)\|^2_{W_2} \to 0, \quad \|f_{2,n}(x) - g_2(x)\|^2_{W_2} \to 0.$$

Let

$$\mathbf{G}(x) = (g_1(x), g_2(x))^T.$$

By Definition 2.1, $\mathbf{G}(x) \in W_{(2,2)}[0, 1]$, and

$$\|\mathbf{F}_n(x) - \mathbf{G}(x)\|^2_{W_{(2,2)}} = \|f_{1,n}(x) - g_1(x)\|^2_{W_2} + \|f_{2,n}(x) - g_2(x)\|^2_{W_2} \to 0.$$

So, the space $W_{(2,2)}[0, 1]$ is a Hilbert space, we call it the reproducing kernel direct space. □