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A numerical scheme for approximating interior jump discontinuity solution of a compressible Stokes system

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ABSTRACT

In this paper we develop a numerical scheme for approximating interior jump discontinuity solutions of compressible Stokes flows with inflow jump datum. The scheme is based on a decomposition of the velocity vector into three parts: the jump part, an auxiliary one and the smoother one. The jump discontinuity is handled by constructing a vector function extending the density jump value of the normal vector on the interface to the whole domain. We show existence of the finite element solutions for the three parts, derive error estimates and also convergence rates based on the piecewise regularities. Numerical examples are given, confirming the derived convergence rates.

1. Introduction

In this paper our concern is a numerical scheme for approximating interior jump discontinuity solutions of the compressible Stokes flows with inflow jump datum. Our scheme is based on a decomposition (see [1, Theorem 1.3]) of the exact solution into three parts: the jump part, an auxiliary one and the smoother one. The jump discontinuity part for the velocity vector is a vector function extending the density jump value on the interface to the whole domain.

To deal with this issue we consider the following simple compressible Stokes system

$$\begin{aligned} -\mu\Delta\mathbf{u}+\nabla p &= \mathbf{f} && \text{in } \Omega,\\ \mathbf{U}\cdot\nabla p+\operatorname{div}\mathbf{u} &= g && \text{in } \Omega,\\ \mathbf{u} &= 0 && \text{on } \Gamma,\\ p &= p_0 && \text{on } \Gamma_{\text{in}}, \end{aligned} \tag{1.1}$$

where $\mathbf{u}$ is the velocity vector and p is the pressure function; μ is the viscosity of the fluid flows with $\mu > 0$; $\mathbf{U} = (1,0)^t$, $\mathbf{f}$, g and p_0 are given data; $\Omega = (0,1)\times(-1,1)$ is the rectangle in the plane $\mathbb{R}^2$, $\Gamma := \partial\Omega$ is the boundary of Ω and $\Gamma_{\text{in}} = \{(0,y)\in\Gamma : -1 < y < 1\}$ is the inflow boundary and $\Gamma_{\text{out}} = \{(1,y)\in\Gamma : -1 < y < 1\}$ is the outflow one. (See Fig. 1.)

It is assumed that the inflow datum p_0 is of the following form having the jump across $y = 0$:

$$p_0(y)\neq 0 \quad \text{for } y\geq 0 \text{ and } 0 \text{ for } y<0. \tag{1.2}$$

Let $\mathcal{C} = \{(x,0)\in\Omega : 0 < x < 1\}$ be the jump curve. It is directed by the vector $\mathbf{U}$, starts at the jump point $(0,0)$ and arrives at the point $(1,0)$. By $\mathcal{C}$ we split the domain Ω into two subregions $\Omega_1 = \{(x,y)\in\Omega : y < 0\}$ and $\Omega_2 = \{(x,y)\in\Omega : y > 0\}$.

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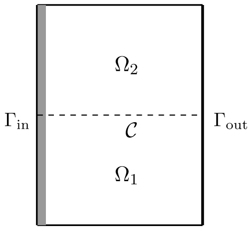

Fig. 1. The domain Ω.

Discontinuities often occur in fluid flows. They are caused by initial shocks, pressure differences or singular boundaries, etc. Hence they are very important issues and also fundamental both in the physical theory of thermodynamics of non-equilibrium states or phase transitions (solid–liquid–gas, vice versa). In particular the density jump on the inflow boundary may result in the jump discontinuities in the interior of the domain (see [2,3]) and the pressure gradient on the momentum equation is not well-defined across the interface that the jump occurs. So we construct a vector function extending the density jump value on the interface to the whole domain and decompose the solution into the jump discontinuity and smoother parts (see [1,4,5]).

In this paper we use the following spaces (see [6]). For any $s \geq 0$ and $q > 1$, $H^{s,q}(\mathcal{O})$ denotes the usual Sobolev space with norm $\|\cdot\|_{s,q,\mathcal{O}}$, where $\mathcal{O}$ is a bounded domain in the plane $\mathbb{R}^2$. If $q = 2$ we write $H^s(\mathcal{O}) = H^{s,q}(\mathcal{O})$ and the norm $\|\cdot\|_{s,\mathcal{O}} = \|\cdot\|_{s,q,\mathcal{O}}$. The space $H^1_0(\mathcal{O}) = \{u \in H^1(\mathcal{O}) : u|_{\partial\mathcal{O}} = 0\}$. We write $H^{-s}(\mathcal{O})$ the dual space of $H_0^{s,q'}$ with norm

$$\|f\|_{-s,q,\mathcal{O}} = \sup_{0\neq v\in H_0^{s,q'}(\mathcal{O})} \frac{\langle f, v\rangle}{\|v\|_{s,q',\mathcal{O}}},$$

where $\langle\ ,\ \rangle$ denotes the duality pairing and q' is the Hölder conjugate of q. Also $L^\infty(\mathcal{O})$ denotes the set of all measurable functions v in $\mathcal{O}$ such that $\|v\|_{\infty,\mathcal{O}} = \operatorname{ess\,sup}\{|v(x)| : x \in \mathcal{O}\}$, and $C^{\alpha,\beta}(\mathcal{O})$ denotes the classical Hölder space with norm $[v]^{\alpha,\beta}$, where α is a nonnegative integer and β is a real number.

Let $Q = \{\eta \in L^2 : \eta_x \in L^2,\ \eta = 0 \text{ on } \Gamma_{\text{in}}\}$ with norm $\|\eta\|_Q = \|\eta\|_0 + \|\eta_x\|_0$. Let $H^1_* = \{v \in H^1 : v = 0 \text{ on } \Omega_1\}$ and $Q^* = \{v \in Q : v = 0 \text{ on } \Omega_1\}$. Let h be the approximation indicator. Let V_h, V_h^*, Q_h and Q_h^* be finite dimensional subspaces of H^1_0, H^1_*, Q and Q^*, respectively. For vector spaces we denote by $\mathbf{V}_h = V_h \times V_h$ and similar for others.

For existence of the problem (1.1) we refer to the reference [1, Remark 1]: If $\mathbf{f} \in \mathbf{H}^{-1}$, $g \in L^2$ and $p_0 \in L^2(\Gamma_{\text{in}})$, there is a unique solution $(\mathbf{u}, p) \in \mathbf{H}^1_0 \times L^2$ of (1.1), satisfying the a priori estimate

$$\mu^{1/2}\|\mathbf{u}\|_1 + \|p\|_0 + \|p_x\|_0 + \|p\|_{0,\Gamma_{\text{out}}} \leq C(\|\mathbf{f}\|_{-1} + \|g\|_0 + \|p_0\|_{0,\Gamma_{\text{in}}}) \tag{1.3}$$

for a constant C.

Basically the usual finite element method for the problem (1.1) is formulated as follows. Based on the weak formulation given in [7] one can find the approximation solution $(\tilde{\mathbf{u}}_h, \tilde{p}_h - p_0) \in \mathbf{V}_h \times Q_h$ for the solution $(\mathbf{u}, p - p_0)$ of (1.1) such that

$$\begin{aligned} &\mu\int_\Omega \nabla\tilde{\mathbf{u}}_h\cdot\nabla\mathbf{v}_h d\mathbf{x} - \int_\Omega \tilde{p}_h \operatorname{div}\mathbf{v}_h d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h d\mathbf{x}, \quad \forall\mathbf{v}_h \in \mathbf{V}_h,\\ &\int_\Omega (\tilde{p}_h - p_0)_x\eta_{h,x} d\mathbf{x} + \int_\Omega \operatorname{div}\tilde{\mathbf{u}}_h\eta_{h,x} d\mathbf{x} = \int_\Omega g\eta_{h,x} d\mathbf{x}, \quad \forall\eta_h \in Q_h. \end{aligned} \tag{1.4}$$

However, with the approximation scheme (1.4) it is not suitable to catch the jump discontinuity behavior due to the jump datum (1.2). To approximate the jump behavior we shall employ a decomposition scheme recently shown in the reference [1]. There the solution was decomposed into the (jump) discontinuous and continuous parts which enables us to get a piecewise regularity of the solution. For readers we summarize the decomposition below. By [1, Theorem 1.1] the solution pair $(\mathbf{u}, p)$ of (1.1) can be split into

$$\mathbf{u} = \mathbf{K} + \mathbf{J} + \mathbf{w}, \qquad p = p_0 + k + j + q \tag{1.5}$$

where the smoother part $(\mathbf{w}, q) \in \mathbf{H}^2 \times H^1$ and solves the following equations:

$$\begin{aligned} &-\mu\Delta\mathbf{w} + \nabla q = \mathbf{h} \quad \text{in } \Omega,\ \ \mathbf{w} = 0 \text{ on } \Gamma,\\ &\qquad q(x, y) = \int_0^x (g(s, y) - \operatorname{div}\mathbf{w}(s, y))ds \end{aligned} \tag{1.6}$$

where $\mathbf{h} = \mathbf{f}$ in Ω_1 and $\mathbf{h} = \mathbf{f} - \nabla p_0 - \nabla k - \nabla j + \mu\Delta\mathbf{K}$ in Ω_2.