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Adaptive generalized multiscale finite element methods for H(curl)-elliptic problems with heterogeneous coefficients

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ABSTRACT

In this paper, we construct an adaptive multiscale method for solving H(curl)-elliptic problems in highly heterogeneous media. Our method is based on the generalized multiscale finite element method. We will first construct a suitable snapshot space, and a dimensional reduction procedure to identify important modes of the solution. We next develop and analyze an a posteriori error indicator, and the corresponding adaptive algorithm. In addition, we will construct a coupled offline–online adaptive algorithm, which provides an adaptive strategy to the selection of offline and online basis functions. Our theory shows that the convergence is robust with respect to the heterogeneities and contrast of the media. We present several numerical results to illustrate the performance of our method.

1. Introduction

Many practical problems are modeled by partial differential equations with highly heterogeneous coefficients. Classical numerical methods for solving these problems typically require very fine computational meshes, and are therefore very expensive to use. In order to solve these problems efficiently, one needs some types of model reduction, which is typically based on upscaling techniques or multiscale methods. In upscaling methods, the heterogeneous coefficient is carefully replaced by an effective medium [1–4] so that the system can be solved on a much coarser grid. In multiscale methods, such as those in [5–17], one attempts to represent the solution by some multiscale basis functions. These basis functions are constructed carefully and are usually based on some local cell problems. The purpose is to capture the fine scale properties of the true solution by using a few multiscale basis functions, with the aim of reducing computational costs.

In this paper, we consider the H(curl)-elliptic problem with highly heterogeneous coefficients. Our aim is to construct a multiscale method for solving this problem. We will consider the generalized multiscale finite element method (GMsFEM) [18,19]. GMsFEM is a generalization of the classical multiscale finite element method [20] in the way that multiple basis functions are used for each coarse region. We will consider three important components of the GMsFEM in this paper. The first one is basis functions construction. This is a process in the offline stage. To find the basis functions, we will construct a set of snapshot functions for each local coarse region. The snapshot functions are solutions of local cell problems with some suitable boundary conditions. To obtain the offline basis functions, we perform a dimension reduction procedure by using a suitable spectral problem, designed carefully based on analysis. These basis functions are then used in a coarse scale conforming finite element formulation to solve the problem. The second component is offline adaptivity [21]. In order to determine the number of offline basis functions to be used for each coarse region, we will develop a local error indicator based on an a posteriori error analysis. Using the proposed error indicator, we are able to determine the number of basis

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functions in an adaptive way. In addition, we prove the convergence of this approach, and show that the convergence rate is independent of the heterogeneities of the coefficients. The last component is online adaptivity [22]. The goal of online basis functions is to capture some components, such as global feature, of the solution that are not representable by offline basis functions. To compute online basis functions, we solve local cell problems by using local residual of the solution. Moreover, we can do this in an adaptive way, so that online basis functions are only added in regions with larger errors. We prove the convergence of the online adaptive method and show that the convergence rate is independent of the coefficients. We also show that a sufficient number of offline basis functions is needed in order to obtain a rapid convergence rate of the online adaptive method. We remark that there are also related methods developed for the discontinuous Galerkin formulation in [23] and [24]. We also remark that a method based on HMM is developed in [25].

To illustrate the performance of our GMsFEM, we present some numerical results focusing on the convergence properties of the method. We will first show that the method is robust with respect to the contrast and heterogeneities of the coefficients. Next, we illustrate the advantage of using offline adaptivity by comparing the convergence behavior with uniform basis enrichment, and show that the offline adaptive method is able to capture the solution more effectively. Finally, we construct a coupled offline–online adaptive method. It is known that the first few offline basis functions correspond to the dominant components of the solution, and the rest of the offline basis functions contribute the solution in a less crucial way. So, one needs to switch to the use of online basis functions once sufficient number of offline basis functions are used. Our offline–online adaptive method allows this to be done automatically. By using a suitable error indicator and a suitable tolerance parameter, we show that the offline–online adaptive method performs very well and give a practical solver for realistic applications.

We summarize the contributions of this paper as follows. First, we developed a set of H(curl)-conforming multiscale vector basis functions. This is based on the proposed local snapshot spaces and the local spectral problems, which are based on our careful analysis. Secondly, a novel offline–online adaptive method is presented. By using an appropriate error indicator, one can switch from offline adaptivity to online adaptivity automatically to reduce the error significantly. Finally, we give an analysis for the convergence of our adaptive method.

The rest of the paper is organized as follows. In the next section, we briefly introduce the basic idea of the GMsFEM. In Section 3, we will present both the offline and online adaptive methods, and in Section 4, we will analyze these methods. In Section 5, numerical results are presented to illustrate the performance of the adaptive methods. Finally, the paper ends with a conclusion.

2. The GMsFEM

In this section, we will give the construction of our GMsFEM for $H(\text{curl})$-elliptic problem. First, we present some basic notations and the coarse grid formulation in Section 2.1. Then, we present the constructions of the multiscale snapshot functions, basis functions and the multiscale scheme in Section 2.2. We will mainly present our ideas in the two-dimensional settings. The extension to the three-dimensional case is straightforward.

2.1. Preliminaries

Let D be a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary ∂D with unit tangential vector t. In this paper, we consider the following high-contrast $H(\text{curl})$-elliptic problem

$$\begin{aligned} \nabla \times (a\, \nabla \times u) + b\, u = f \qquad & \text{in } D, \\ u \cdot t = 0 \qquad & \text{on } \partial D, \end{aligned} \tag{1}$$

where $a \geq 1$ is a heterogeneous field with high contrast, $b > 0$ is a bounded heterogeneous field and f is a given divergence-free source.

To describe the general solution framework for the model problem (1), we first introduce the notion of fine and coarse grids. Let $\mathcal{T}^h$ be a partition of the domain D into fine finite elements. Here $h > 0$ is the fine mesh size. The coarse partition, $\mathcal{T}^H$ of the domain D, is formed such that each element in $\mathcal{T}^H$ is a connected union of fine-grid blocks. More precisely, $\forall K_j \in \mathcal{T}^H,\ K_j = \cup_{F \in I_j} F$ for some $I_j \subset \mathcal{T}^h$. The quantity $H > 0$ is the coarse mesh size. We will consider rectangular coarse elements and the methodology can be used with general coarse elements. An illustration of the mesh notations is shown in Fig. 1(Left).

Next, we define the finite element space V_{h} as the set of the lowest order curl conforming elements of Nédélec with respect to the fine mesh $\mathcal{T}^h$, and define $V_{\text{h}}^0 = \{v \in V_{\text{h}} \mid v \cdot t = 0 \text{ on } \partial D\}$. The fine-scale solution $u_{\text{h}} \in V_{\text{h}}$ is obtained by solving the following variational problem

$$\int_D \Big(a\, (\nabla \times u_{\text{h}}) (\nabla \times v) + b\, u_{\text{h}} \cdot v \Big) = \int_D f \cdot v \quad \forall v \in V_{\text{h}}^0. \tag{2}$$

The solution u_{h} is our reference solution. The convergence property of this method is well-known (see for example [26]).

Finally, for any subdomain $\Omega \subset D$, and $v \in V_{\text{h}}$, we define the norms $\|v\|_{L^2(b;\Omega)}$ and $\|v\|_{H(\text{curl})(a,b;\Omega)}$ as

$$\|v\|^2_{L^2(b;\Omega)} = \int_\Omega b\, |v|^2$$