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An efficient approach for solving stiff nonlinear boundary value problems

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ABSTRACT

A new method for solving stiff two-point boundary value problems is described and compared to other known approaches using the Troesch's problem as a test example. The method is based on the general idea of alternate approximation of either the unknown function or its inverse and has a genuine "immunity" towards numerical difficulties invoked by the rapid variation (stiffness) of the unknown solution. A c++ implementation of the proposed method is available at https://github.com/imathsoft/MathSoftDevelopment.

1. Introduction

In the present paper we consider a nonlinear boundary value problem (BVP)

$$\frac{d^2u(x)}{dx^2} = N\left(u(x), x\right) u(x), \quad x \in [a, b]\,,\; N(u, x) \in C^2(\mathbb{R} \times [a, b]), \tag{1}$$

$$u(a) = u_a \in \mathbb{R}, \quad u(b) = u_b \in \mathbb{R}, \tag{2}$$

which arises in many areas of physics and mathematics. Although, there is a huge variety of known methods for solving problems of type (1), (2) (see, for example [1–4] and the references therein), almost none of them fill comfortable when the problem turns out to be stiff.

As it was pointed out in [5], a good mathematical definition of the concept of stiffness does not exist. The famous definition given in [6] says that "stiff equations are problems for which explicit methods don't work", which, unfortunately, is not very constructive. According to [7], there is at least 6 different definitions of stiff problems which possess different levels of formality and are accepted by different schools of mathematics. The authors of [7] came up with their own definition of "stiffness", based on the concept of *stiffness ratio*, which encompasses all the known definitions.

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In the present paper we confine ourselves to consider only a subclass of stiff boundary value problems (1), (2) whose stiffness is originated from the fact that the exact solution $u(x)$ possesses narrow intervals of rapid variation, known as the *boundary layers*. Such a behavior is typical for *singularly perturbed problems*, which are an important subclass of stiff problems (see, [8–13,7]). The rapid variation is equivalent to having $|u'(x)| \gg 1$ on some subset of $[a, b]$. And it is the need to approximate the solution on this subset that makes the problem numerically difficult and unstable, i.e. stiff. Now to approximate the solution on the subset of $[a, b]$ where $|u'(x)|$ is comparatively small is much easier from the numerical point of view. To be more specific, let us consider a set $\chi_u \in [a, b]$ defined in the following way:

$$\chi_u = \{x \in [a, b] : |u'(x)| \geq 1\}. \tag{3}$$

It is easy to see that, defined in such a way, set χ_u consists of a finite or infinite number of distinctive closed intervals $\bar{\iota}_i$. Some of the intervals $\bar{\iota}_i$ might be those of rapid variation for the solution $u(x)$. At the same time, by the definition of χ_u (3), solution $u(x)$ is strictly monotonic on each interval $\bar{\iota}_i$, which means that we can consider the inverse function $x_{\bar{\iota}_i}(\cdot) = u^{-1}(\cdot)$ defined on the closed interval $u(\bar{\iota}_i) \in u([a, b])$. There are two remarkable things about the function $x_{\bar{\iota}_i}(\cdot)$:

1. $|x'_{\bar{\iota}_i}(u)| \leq 1, \forall u \in u(\bar{\iota}_i)$, which means that the initial BVP stated in terms of "inverse solution" $x'_{\bar{\iota}_i}(u)$ is not stiff on $u(\bar{\iota}_i)$;
2. having function $x_{\bar{\iota}_i}(u)$ approximated on a discrete set of points from $u(\bar{\iota}_i)$ we automatically get function $u(x)$ approximated on some discrete set of points from $\bar{\iota}_i$.

The two observations give us the key insight on how to deal with the subclass of stiff problems defined above. It is the *divide and conquer* principle: on the subintervals where solution $u(x)$ is well behaved (showing rather moderate variation) we solve the given problem (1), (2), whereas on the subintervals $\bar{\iota}_i$, where $u(x)$ varies rapidly (and the initial problem is stiff), we solve the corresponding problem for the inverse solution $x_{\bar{\iota}_i}(u)$. Of course, this becomes feasible from the practical point of view only if there is a finite number of subintervals $\bar{\iota}_i$, which becomes our assumption from now on.

Speaking about the known methods for solving BVPs, it is impossible not to mention the *simple shooting method* (SSM) and the *multiple shooting method* (MSM) [14, Section 7.3] which are two the most simple and reliable techniques to deal with boundary value problems of type (1), (2). By calling them *techniques* and not just *methods* we would like to emphasize that the basic idea behind them is very broad and can be used in many different modifications, which, in turn, might be called *methods*. Since definitions of both SSM and MSM essentially relay on using methods for solving *initial value problems* (IVP), one of the ways to come up with a new modification consists in using a different IVP solver. Below we adapt (modify) the SSM and MSM by using a specific approach for numerical solution of IVP's which is based on the idea of alternate approximation of either straight $u(x)$ or inverse $x(u)$ solutions of Eq. (1) and has a genuine "immunity" towards numerical difficulties invoked by the rapid variation (stiffness) of the solution in question.

The main focus of the paper is not only to present a general idea about how to treat some subclass of stiff boundary value problems in an efficient way, but also to describe and examine a possible particular implementation of the idea, hereinafter referred to as *Straight-Inverse method* (or, simply, SI-method). With this in mind, we actively exploit one of the most famous examples of stiff BVPs, known as the Troesch's problem:

$$\frac{d^2u(x)}{dx^2} = \lambda \sinh\left(\lambda u(x)\right), \quad x \in [0, 1] \tag{4}$$

$$u(0) = 0, \quad u(1) = 1, \tag{5}$$

which is a partial case of problem (1), (2) with $N(u(x), x) \equiv \lambda \sinh\left(\lambda u(x)\right)/u(x)$, $a = u_a = 0$, $b = u_b = 1$. In addition to its application in physics of plasma, the Troesch's problem, has drawn a lot of interest to itself as a test case for methods of solving unstable two-point boundary value problems because of its difficulties [15]. A vast amount of numerical data available for the problem (see [15,7,16–19] and the references therein) allowed us to perform broad analysis of the SI-method and compare it to many other methods for solving two-point BVPs. The comparison confirms excellent characteristics of the method in terms of both accuracy (numerical stability) and performance. Results of multiple numerical tests with problems other than (4), (5) (among them those with $N'_x(u, x) \not\equiv 0$ and with the solution $u(x)$ oscillating on $[a, b]$), which are not included in the present paper, show remarkable adaptivity potential of the SI-method, and do support the conclusions obtained on the Troesch's test problem.

At this point, we would like to notice that, in general case, there is no guarantee that the BVP (1), (2) is solvable, i.e. has a solution. From [20, Theorem 7.25] it follows, however, that, under the conditions imposed on the nonlinearity of Eq. (1), the problem can have at most one solution, that is, the uniqueness is granted. The question of existence is kept out of the scope of the current paper, as well as the error analysis for the SI-method applied to BVP (1), (2). We leave both issues for the future publications. The main theoretical result of the paper, Theorems 1, 2, deals with the SI-method for Eq. (1) subjected to an initial condition, and provides a priori error estimates for the case.

The paper is organized as follows. In the beginning of Section 2 we introduce the SI-method for solving initial value problems associated with Eq. (1); the rest part of the section is devoted to a thorough investigation of the method's approximation properties, which are formulated as Theorems 1 and 2. The SI-method for solving boundary value problem (1) (2) is the main focus of Section 3, where we describe a *single* and *multiple shooting* versions of the method. We apply the SI-method to the Troesch's equation subjected to both initial and boundary value conditions and discuss the results in Section 4. Section 5 contains conclusions.