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Multiscale analysis of heterogeneous domain decomposition methods for time-dependent advection–reaction–diffusion problems

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ABSTRACT

Domain decomposition methods which use different models in different subdomains are called heterogeneous domain decomposition methods. We are interested here in the case where there is an accurate but expensive model one should use in the entire domain, but for computational savings we want to use a cheaper model in parts of the domain where expensive features of the accurate model can be neglected. For the model problem of a time dependent advection–reaction–diffusion equation in one spatial dimension, we study approximate solutions of three different heterogeneous domain decomposition methods with pure advection reaction approximation in parts of the domain. Using for the first time a multiscale analysis to compare the approximate solutions to the solution of the accurate expensive model in the entire domain, we show that a recent heterogeneous domain decomposition method based on factorization of the underlying differential operator has better approximation properties than more classical variational or non-variational heterogeneous domain decomposition methods. We show with numerical experiments in two spatial dimensions that the performance of the algorithms we study is well predicted by our one dimensional multiscale analysis, and that our theoretical results can serve as a guideline to compare the expected accuracy of heterogeneous domain decomposition methods already for moderate values of the viscosity.

1. Introduction

Heterogeneous domain decomposition methods are domain decomposition methods where different models are solved in different subdomains. Models can be different because problems are heterogeneous, i.e. there are connected components with different physical properties, see for example [1–4], or because one wants to approximate a homogeneous object with different approximations, depending on their validity and cost, see for example [5–12]. In this second situation, there is in general a complex, expensive model which would give the best possible solution, and the heterogeneous domain decomposition methods try to give a good approximation to this best possible solution at a lower computational cost. It is therefore possible in this second situation to quantify the quality of heterogeneous domain decomposition approximations in a rigorous mathematical way, by comparing them to the expensive solution on the entire domain, as it was proposed in [13], see also the earlier publication [14]. Using for the first time multiscale analysis, we compare in this paper three heterogeneous

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domain decomposition methods to solve time dependent advection–reaction–diffusion equations, with advection reaction approximations in parts of the domain: the method using variational and non-variational coupling conditions from [15,16], see also [17] and [18], and the factorization method, which has its roots in [17], but was only fully developed in [19] for one dimensional steady advection–reaction–diffusion problems. It was proved in [19] that the factorization method can give approximate solutions in the viscous region which can be exponentially close to the monodomain viscous solution for one dimensional steady problems. A factorization method for time dependent advection–reaction–diffusion problems was proposed in [20], and its performance was studied using a priori error estimates. We present here for the first time a multiscale analysis of the factorization method, together with the variational and non-variational ones, and we show with numerical experiments that the results of this multiscale analysis also describe the behavior of the coupling algorithms very well in higher spatial dimensions.

We present in Section 2 the three heterogeneous domain decomposition methods we will study in this paper for time dependent advection–reaction–diffusion problems. In Section 3, we perform a multiscale analysis of the factorization method, and give sharp error estimates as the viscosity goes to zero. In Section 4, we present the corresponding multiscale analysis for the variational heterogeneous domain decomposition method, and in Section 5 the one for the non-variational heterogeneous domain decomposition method. The error estimates we obtain allow us to compare the quality of the coupled solutions obtained by these three methods, and the results differ, depending on the advection direction at the interface. We then test in Section 6 the three heterogeneous domain decomposition algorithms numerically in a two dimensional setting that goes beyond our theoretical analysis. Our results show that the one dimensional multiscale analysis predicts nevertheless the performance very well also in two dimensions, and this already for moderate values of the viscosity parameter. Our theoretical results are thus really useful to guide people in the choice of coupling conditions for heterogeneous domain decomposition. We finally compare the numerical cost of the algorithms and summarize our findings in Section 7.

2. Heterogeneous domain decomposition methods

We define the time dependent advection–reaction–diffusion operator $\mathcal{L}_{ad} := \partial_t - \nu\partial_x^2 + a\partial_x + c$, $\nu > 0$ and $c \geq 0$, its non-diffusive approximation $\mathcal{L}_a := \partial_t + a\partial_x + c$, and consider two model problems: for positive advection $a > 0$, we want to approximate

$$\begin{array}{rcll} \mathcal{L}_{ad}u & = & f & \text{in } (-L_1, L_2) \times (0, T), \\ u(-L_1, \cdot) & = & g_1 & \text{on } (0, T), \\ \mathcal{L}_a u(L_2, \cdot) & = & 0 & \text{on } (0, T), \\ u(\cdot, 0) & = & h & \text{in } (-L_1, L_2), \end{array} \tag{2.1}$$

which represents the outflow from a region where viscosity is important into an area where it is not. The boundary condition at outflow is absorbing, see [21]. For negative advection, $a < 0$, we want to approximate

$$\begin{array}{rcll} \mathcal{L}_{ad}u & = & f & \text{in } (-L_1, L_2) \times (0, T), \\ u(-L_1, \cdot) & = & g_1 & \text{on } (0, T), \\ u(L_2, \cdot) & = & g_2 & \text{on } (0, T), \\ u(\cdot, 0) & = & h & \text{in } (-L_1, L_2), \end{array} \tag{2.2}$$

which represents the inflow from a region where the viscosity is not important into an area where it is, i.e. a boundary layer which is forming on the left. In both model problems (2.1) and (2.2), we want to approximate the solution by solving an advection–reaction–diffusion equation in the domain $\Omega_1 := (-L_1, 0)$, and only an advection reaction equation in $\Omega_2 := (0, L_2)$.

2.1. Variational coupling conditions

A heterogeneous domain decomposition method using variational coupling conditions was introduced in [15,16] for stationary problems. The method was obtained in a variational framework, by fixing the viscosity in a subregion, and then letting the viscosity go to zero in the remaining domain. The method is non-iterative, and when extended to our time dependent setting, it consists for $a > 0$ in solving first the advection–reaction–diffusion problem

$$\begin{array}{rcll} \mathcal{L}_{ad}u_{ad}^V & = & f & \text{in } \Omega_1 \times (0, T), \\ u_{ad}^V(-L_1, \cdot) & = & g_1 & \text{on } (0, T), \\ \partial_x u_{ad}^V(0, \cdot) & = & 0 & \text{on } (0, T), \\ u_{ad}^V(\cdot, 0) & = & h & \text{in } \Omega_1, \end{array} \tag{2.3}$$

followed by solving the advection reaction problem

$$\begin{array}{rcll} \mathcal{L}_a u_a^V & = & f & \text{in } \Omega_2 \times (0, T), \\ u_a^V(0, \cdot) & = & u_{ad}^V(0, \cdot) & \text{on } (0, T), \\ u_a^V(\cdot, 0) & = & h & \text{in } \Omega_2. \end{array} \tag{2.4}$$

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