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DG framework for pricing European options under one-factor stochastic volatility models

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ABSTRACT

The modern theory of option pricing is based on models introduced almost 50 years ago. These models, however, are not able to capture real market behaviour sufficiently well. One line of extensions consists of introducing an additional variable into the model, the so-called stochastic volatility. Since such models lead to the (semi) closed-form solution only rarely, some form of a numerical approximation can be essential. In this paper we study a general one-factor stochastic volatility model for the pricing of European options. A standard mathematical approach to this problem leads to a degenerate partial differential equation completed by boundary and terminal conditions. We formulate this problem in a variational sense and prove the existence and the uniqueness of a weak solution. Further, a robust numerical procedure based on the discontinuous Galerkin approach is proposed to improve the numerical valuation process. The performance of the procedure is accompanied with theoretical results and documented using reference experiments with the emphasis on investigation of the behaviour of option values with respect to the different mesh sizes as well as polynomial orders of approximation.

1. Introduction

The modern theory of option pricing goes back to the ideas of Black and Scholes (BS), independently published in [1] and [2]. Nowadays, it is widely accepted that the BS model is not sufficiently accurate in capturing the real world features of security markets, because its idealized assumptions do rarely hold in practice. Among them, the most serious are the requirements on the Gaussianity of asset log-returns and their constant volatility. Despite its obvious drawbacks, the BS model is widely used in practice, though the constant volatility is replaced by an artificial quantity called implicit volatility, value of which allows one to match the theoretical price according to the BS model with the market price.

However, by relaxing even one of the assumptions mentioned above, one can get advanced option pricing models that show much better performance and there is no need for the artificial procedure of implicit volatility. For example, the constant parameter of volatility can easily be replaced by an additional stochastic process (i.e., the so-called stochastic volatility models) as already suggested by, e.g., Hull and White [3], Scott [4] and Wiggins [5] during late 1980s.

At present, there exists a wide class of stochastic volatility models, see [6] and references cited therein. A standard mathematical approach brings a problem described by a (degenerate) partial differential equation (PDE) completed by boundary and terminal (initial) conditions. Notwithstanding, such system leads to (semi) closed-form solutions rarely and under specific conditions only. Hence, one has to rely on various numerical approximations.

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Table 1
The frequently used stochastic volatility models. The non-negative constants α, β and m are usually called the mean reversion rate, the volatility of volatility and the long term mean of Y.

Model	$f(Y)$	$A(Y)$	$B(Y)$
Hull–White [3]	$\sqrt{Y}$	αY	βY
Scott [4]	$\exp(Y)$	$\alpha(m - Y)$	β
Stein–Stein [14]	$\lvert Y \rvert$	$\alpha(m - Y)$	β
Heston [15]	$\sqrt{Y}$	$\alpha(m - Y)$	$\beta\sqrt{Y}$

The performance demand on the numerical valuation process is very high and several techniques have been developed during last decades to obtain efficient pricing algorithms, including lattice/trees methods [7], finite difference schemes [8] and finite element approaches [9]. Obviously, also these methods have their own limitations in the treatment of numerical option pricing under more complex market conditions.

In this work we provide deeper insight into one of the robust and accurate numerical methods. We propose a numerical technique based on the discontinuous Galerkin (DG) method (see [10]) so that the option pricing procedure under a wide spectrum of volatility models with various driving processes can be unified. Our motivation for the discontinuous approach, in comparison with the standard techniques, is based on the idea that this alternative view of solving the problem considered enables us to resolve option values more properly, e.g., a piecewise linear payoff function has a discontinuous derivative or discrete barriers can be easily implemented. Note that the DG method does not represent a completely new approach to the issue of option valuation, see recent studies devoted to pricing of simple vanilla [11], Asian [12] and American [13] options.

In this paper, we proceed as follows. In Section 2 a general one-factor volatility model is introduced. Next, in Section 3 a robust discontinuous Galerkin framework is proposed and subsequently in Section 4 numerical experiments are provided with the emphasis on the behaviour of the option values with respect to the discretization parameters.

2. One-factor stochastic volatility models

Stochastic volatility models are a variance extension of the classical Black–Scholes model dynamics by introducing another auxiliary processes to model the volatility of the underlying asset returns. More precisely, we generally speak of multi-factor (or multi-scale) stochastic volatility models. The simplest class of them is formed by the one-factor models, where the volatility is determined by one driving process only. Despite that, such tool represents a powerful modification of the BS model that considers a much more complex market conditions.

The forthcoming section is organized as follows. We start with a description of the model dynamics and the derivation of the corresponding pricing equation for European options. Then we formulate the problem in a variational sense and prove the existence and the uniqueness of a weak solution.

2.1. Model dynamics

Let us consider a financial asset S whose price at time t is given by the stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t, \tag{1}$$

where $\mu S_t dt$ is a drift term with a constant rate μ, W_t is a standard Brownian motion and σ_t is the volatility, see [2]. In one-factor stochastic volatility models, we assume that $\sigma_t = f(Y_t)$ is a function of a general Markovian Itô driving process

$$dY_t = A(Y_t)dt + B(Y_t)dZ_t, \tag{2}$$

where Z_t is a second Wiener process correlated to W_t with a factor $\rho \in (-1, 1)$. The mappings f, A and B are real functions defined on the interval $I = \mathbb{R}$ or $I = \mathbb{R}^+$, and their regularity will be specified later, see Sections 2.3 and 3.2. Their specific forms include the different types of the driving processes and define the particular stochastic volatility models. Obviously, the functions f and B have to be non-negative. The frequently used models, included in our study, are listed in Table 1. While the Hull–White model assumes that the volatility is driven by a lognormal process, the Scott and Stein–Stein models use a mean-reverting Ornstein–Uhlenbeck (OU) process and the Heston dynamics is described by the Cox–Ingersoll–Ross (CIR) process.

Remark 1. Note that f, A and B can depend also on the underlying asset S and time t in general, then one speaks of the local-stochastic volatility models. Here, for simplicity, we consider only Y-dependence, i.e., $f = f(Y)$, $A = A(Y)$ and $B = B(Y)$.

2.2. Derivation of the pricing equation

Consider a European option on a financial asset S (for the sake of clarity we omit the subscript t in the rest of the paper) with maturity T and assume the instantaneous risk-free interest rate r. The option price $V(S, Y, t)$ depends on the underlying asset S, the driving process Y and the actual time t.