Journal of Computational and Applied Mathematics 344 (2018) 173–189

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Relaxation model for the p-Laplacian problem with stiffness

Heesun Choi [a], Hongjoong Kim [a,*], Marc Laforest [b]

[a] Department of Mathematics, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul, 02841, Republic of Korea
[b] Département de mathématiques et de génie industriel, École Polytechnique de Montréal, Québec, H3C 3A7, Canada

ARTICLE INFO

Article history:
Received 6 July 2015
Received in revised form 28 February 2018

Keywords:
Degenerate parabolic
Relaxation scheme
p-Laplacian
Hyperbolic system
Superconductivity
Power-law model

ABSTRACT

This paper proposes a new numerical scheme in 1-D for the p-Laplacian problem for the electromagnetic effects in a high-temperature Type II superconductors. The scheme is obtained by applying a relaxation approximation to the nonlinear derivatives in the problem. The new relaxation scheme achieves highly accurate results even for large p that makes the p-Laplacian flux stiff. The scheme is novel in that it is high-order accurate and predicts physically correct non-oscillatory magnetic fronts within these conductors, the later of which is not found by finite element approximate solutions done by the engineering community. The work is an extension of previous work on relaxation schemes applied to degenerate parabolic problems. Numerical tests are presented to validate the performance of the new scheme.

1. Introduction

In this paper we establish a new numerical scheme for the nonlinear parabolic system (1.1) called the p-curl system [1] for the electromagnetic effects in a high-temperature superconductor (HTS):

$$\partial_t \mathbf{B} + \nabla \times \left(\rho(\nabla \times \mu^{-1}\mathbf{B})\nabla \times \mu^{-1}\mathbf{B}\right) = \mathbf{Q}(\boldsymbol{x}, t)\,, \tag{1.1a}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{1.1b}$$

where $\mathbf{B}$ is a magnetic flux density in $\mathbb{R}^3$, μ is the magnetic permeability, and ρ is the magnetic resistivity. The divergence constraint (1.1b) makes the $\nabla \times \nabla \times$ operator coercive [2], and in the nonlinear case, monotone [3,4]. Hence for divergence free initial conditions and source terms $\mathbf{Q}$, and consistent boundary conditions the system (1.1) has a unique stable solution. This paper focuses exclusively on the one dimensional problem where the divergence constraint is always satisfied and the p-curl reduces to the better-known p-Laplacian problem.

It is known that the p-Laplacian system is challenging to solve because of the nonlinearity and stiffness introduced by the resistivity function $\rho = \rho(\nabla \times \mu^{-1}\mathbf{B})$. Here we only consider the resistivity of the power law model [5]

$$\rho(\mathbf{J}) = \frac{E_0}{J_c}\left|\frac{\mathbf{J}}{J_c}\right|^{p-2} \tag{1.2}$$

where E_0, J_c and p are some constants that depend on physical properties of the superconductor. As $p \to \infty$, this model and its solutions converge to those of the discontinuous Bean model [6,7]. In practice, p takes large values between 10 and 100 which can be determined experimentally for Type II HTS.

* Corresponding author.
E-mail addresses: heesun_choi@korea.ac.kr (H. Choi), hongjoong@korea.ac.kr (H. Kim), marc.laforest@polymtl.ca (M. Laforest).

https://doi.org/10.1016/j.cam.2018.05.022
0377-0427/

Relaxation approximations were first analyzed by Liu in his 1987 paper [8] in which he introduced the *sub-characteristic* stability condition. Numerical schemes were later developed from his idea, notably following the influential papers of Jin and Xin [9] and of Jin and Levermore [10]. In [9,10], for a scalar conservation law

$$\partial_t u + \partial_x f(u) = s(u), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \ \ u \in \mathbb{R}, \tag{1.3}$$

a so-called *relaxation scheme* is introduced:

$$\begin{aligned} &\partial_t u + \partial_x v = s(u), \\ &\partial_t v + a\, \partial_x u = -\frac{1}{\epsilon}(v - f(u)), \end{aligned} \tag{1.4}$$

where ϵ is a small positive parameter and a is a fixed positive constant. The scheme (1.4) is proved to achieve high order accuracy without using Riemann solvers or a nonlinear system of algebraic equations solvers.

Naldi et al. [11–13] extended the ideas of Jin and Xin [9] and applied it to a nonlinear degenerate parabolic equation of the form

$$\partial_t u + \partial_x f(u) = \partial_{xx} h(u) + s(u), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \ \ u \in \mathbb{R}, \tag{1.5}$$

where $h(u)$ is a smooth function such that $h(0) \geq 0$ and $h'(u) > 0$ to propose a relaxation scheme :

$$\begin{aligned} &\partial_t u + \partial_x v = s(u), \\ &\partial_t v + a\, \partial_x w = -\frac{1}{\epsilon}(v - f_\epsilon(u, \partial_x w)), \\ &\partial_t w + \partial_x v = -\frac{1}{\epsilon}(w - h(u)). \end{aligned} \tag{1.6}$$

where $\epsilon > 0$, $a \in (0, 1/\epsilon)$, and $f_\epsilon(u, \partial_x w) = f(u) - (1 - \epsilon a)\partial_x w$. The scheme (1.6) replaces the second order diffusion equations with first order semi-linear hyperbolic systems with stiff terms.

The work presented in this paper is inspired by the relaxation schemes (1.4) and (1.6). Both schemes are applicable to the p-Laplacian system and they give good results for simple examples with small p. We adapt the schemes (1.4) and (1.6) in order to construct a new relaxation scheme for the p-Laplacian system (1.1). The new scheme is distinguished by its simplicity, its high-order accuracy and the fact that the solutions are non-oscillatory. The last two properties are not found in current finite element approximations done by the engineering community. It maintains high order of accuracy even for an example with a stiff flux.

The rest of the paper is organized as follows. Section 2 compares the applications of relaxations (1.4) and (1.6) for a porous media equation and shows the scheme (1.6) results in less diffusion and higher accuracy. However an integration is required in applying the scheme (1.6) to the p-Laplacian problem (1.1) and it reduces the accuracy even with various well-known integration methods. In Section 3, we propose a new relaxation scheme which does not require an integration. Specific numerical discretizations and error estimations are described. Section 4 presents the numerical results of the new scheme. Section 5 concludes the results and suggests some future avenues of research.

We finish this section with a remark that the positive integer p is limited to an even number throughout this paper, for simplicity.

2. Relaxation approximations for p-Laplacian problem

Let us consider the one dimensional case of the p-Laplacian equation (1.1) where the **B** field propagates in the x direction while oscillating in the y direction. In this case, all components of **B** except the second are zero, i.e.,

$$\mathbf{B}(\boldsymbol{x}) = (0, B(x), 0),$$

and the divergence constraint (1.1b) is automatically satisfied. In (1.1a) we assume the parameter μ is a constant, although it can be discontinuous at material interfaces in a practical example. Then, following the definition of resistivity (1.2), Eq. (1.1) becomes

$$\partial_t B + \partial_x f(\partial_x B) = Q(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \tag{2.1}$$

with flux defined by

$$f(\partial_x B) := -c_0(\partial_x B)^{p-1}\,, \tag{2.2}$$

where $c_0 = E_0/(\mu J_c)^{p-1}$. For simplicity, let $c_0 = 1$. The well-posedness theory of the p-Laplacian is well-established with existence in $H^1(\mathbb{R}, L^p(\mathbb{R})) \cap L^\infty(\mathbb{R}, W^{1,p}(\mathbb{R}))$ but throughout, we will simply assume the solutions are sufficiently smooth to warrant the finite differences that are applied.

Jin and Xin [9] suggest a relaxation system (1.4) to solve a conservation law (1.3). It aims to replace the original system (1.3) with a semi-linear form and provide a way to derive simple and Riemann solver free method. Since, in [9], the relaxation