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High dimensional integration of kinks and jumps—Smoothing by preintegration

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ABSTRACT

We show how simple kinks and jumps of otherwise smooth integrands over $\mathbb{R}^d$ can be dealt with by a preliminary integration with respect to a single well chosen variable. It is assumed that this preintegration, or conditional sampling, can be carried out with negligible error, which is the case in particular for option pricing problems. It is proven that under appropriate conditions the preintegrated function of $d-1$ variables belongs to appropriate mixed Sobolev spaces, so potentially allowing high efficiency of Quasi Monte Carlo and Sparse Grid Methods applied to the preintegrated problem. The efficiency of applying Quasi Monte Carlo to the preintegrated function are demonstrated on a digital Asian option using the Principal Component Analysis factorization of the covariance matrix.

1. Introduction

In the present paper we analyze a natural method for numerical integration over $\mathbb{R}^d$, where d may be large, in the presence of "kinks" (i.e. discontinuities in the gradients) or "jumps" (i.e. discontinuities in the function). In this method one of the variables is integrated out in a "preintegration" step, with the aim of creating a smooth integrand over $\mathbb{R}^{d-1}$.

Integrands with kinks and jumps arise in option pricing, because an option is normally considered worthless if the value falls below a predetermined strike price. In the case of a continuous payoff function this introduces a kink, while in the case of a binary or other digital option it introduces a jump.

A simple strategy is to ignore the kinks and jumps, and apply directly a method for integration over $\mathbb{R}^d$. While there has been very significant recent progress in *Quasi Monte Carlo (QMC) methods* [1] and *Sparse Grid (SG) methods* [2] for high dimensional integration when the integrand is somewhat smooth, there has been little progress in understanding their performance when the integrand has kinks or jumps.

The performance of QMC and SG methods is degraded in the presence of kinks and jumps, but perhaps not as much as might have been expected, given that in both cases the standard error analysis fails in general for kinks and jumps: the standard assumption in both cases is that the integrand has mixed first partial derivatives for all variables, or at least that it has bounded Hardy and Krause variation over the unit cube $[0, 1]^d$, whereas even a straight non-aligned kink (one that is not orthogonal to one of the axes) lacks mixed first partial derivatives even for $d = 2$, and generally exhibits unbounded Hardy and Krause variation on $[0, 1]^d$ for $d \geq 3$ [3].

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A possible path towards understanding the performance of QMC and SG methods in the presence of kinks and jumps was developed in [4]. That paper studied the terms of the "ANOVA decomposition" of functions with kinks defined on d-dimensional Euclidean space $\mathbb{R}^d$, and showed that under suitable circumstances all but one of the 2^d ANOVA terms can be smooth, with the single exception of the highest order ANOVA term, the one depending on all d of the variables. If the "effective dimension" of the function is small, as is commonly thought to be the case in applications, then that single non-smooth term can be expected to make a very small contribution to both supremum and $\mathcal{L}_2$ norms. In a subsequent paper [5] the same authors showed, by strengthening the theorems and correcting a mis-statement in [4], that the smoothing of all but the highest order ANOVA term is a reality for the case of an arithmetic Asian option with the Brownian bridge construction.

More precisely, the papers [4] and [5] showed, for a function of the form $f(\boldsymbol{x}) = \max(\phi(\boldsymbol{x}), 0)$ with ϕ smooth (so that f generically has a kink along the manifold $\phi(\boldsymbol{x}) = 0$), that if the d-dimensional function ϕ has a positive partial derivative with respect to x_j for some $j \in \{1, \ldots, d\}$, and if certain growth conditions at infinity are satisfied, then all the ANOVA terms of f that do not depend on the variable x_j are smooth. The underlying reason, as explained in [4], is that integration of f with respect to x_j, under the stated conditions, results in a $(d-1)$-dimensional function that no longer has a kink, and indeed is as often differentiable as the function ϕ.

Going beyond kinks, we prove in this paper that Theorem 1 in [5] can be extended from kinks to jumps—thus jumps are smoothed under almost the same conditions as kinks. The smoothing occurs even in situations (for example in option pricing) where the location of the kink or jump treated as a function of the other $d-1$ variables moves off to infinity for some values of the other variables.

In this paper we pay particular attention to proving that the presmoothed integrand belongs to an appropriate mixed-derivative function space.

The preintegration method studied in the present paper has appeared as a practical tool under other names in many other papers, including those related to "conditional sampling" (see [6]; the paragraph leading up to and including Lemma 7.2 in [7]; the remark at the end of Section 3 in [8]), and other root-finding strategies for identifying where the payoff is positive (see [9,10]), as well as those under the name "smoothing" (see [11,12]). In contrast to the cited papers, the emphasis in this paper is on rigorous analysis. Also, we here prefer the description "preintegration" because to us "conditional sampling" suggests a probabilistic setting, which is not necessarily relevant here.

Even for the classical *Monte Carlo (MC) method* the preintegration step can be useful: to the extent that the preintegration can be considered exact, there is a reduction in the variance of the integrand, by the sum of the variances of all ANOVA terms that involve the preintegration variable x_j (since the ANOVA terms are eliminated because their exact integrals with respect to x_j are all zero). In our numerical experiments that reduction proves to be quite significant.

The problem class and the method are stated in Section 2. Section 3 gives numerical examples in the context of an option pricing problem with 256 time steps, treated as a problem of integration in 256 dimensions. Section 4 briefly discusses the variance reduction by preintegration for $\mathcal{L}_2$ functions. Section 5 focuses on the smoothing effect of preintegration. It gives mathematical background on needed function spaces and states two new smoothing theorems, extended here in a non-trivial way from [5, Theorem 1]. Section 6 applies our theoretical results to the option pricing example. Technical proofs are given in Section 7.

2. The problem and the method

The problem is the approximate evaluation of

$$I_d f := \int_{\mathbb{R}^d} f(\boldsymbol{x})\rho_d(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_d)\,\rho_d(\boldsymbol{x})\,\mathrm{d}x_1 \cdots \mathrm{d}x_d, \tag{1}$$

with

$$\rho_d(\boldsymbol{x}) := \prod_{k=1}^{d} \rho(x_k),$$

where ρ is a continuous and strictly positive probability density function on $\mathbb{R}$ with some smoothness, and f is a real-valued function integrable with respect to ρ_d.

To allow for both kinks and jumps we assume that the integrand is of the form

$$f(\boldsymbol{x}) = \theta(\boldsymbol{x})\,\mathrm{ind}(\phi(\boldsymbol{x})), \tag{2}$$

where θ and ϕ are somewhat smooth functions, and $\mathrm{ind}(\cdot)$ is the indicator function which gives the value 1 if the input is positive and 0 otherwise. When $\theta = \phi$ we have $f(\boldsymbol{x}) = \max(\phi(\boldsymbol{x}), 0)$ and thus we have the familiar kink seen in option pricing through the occurrence of a strike price. When θ and ϕ are different (for example, when $\theta(\boldsymbol{x}) = 1$) we have a structure that includes binary digital options.

Our key assumption on $\phi(\boldsymbol{x})$ is that it has a positive partial derivative (and so is an increasing function) with respect to some variable x_j, that is, we assume that for some $j \in \{1, \ldots, d\}$ we have

$$\frac{\partial \phi}{\partial x_j}(\boldsymbol{x}) > 0 \qquad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{3}$$

In other words ϕ is monotone with respect to x_j.