Journal of Computational and Applied Mathematics 344 (2018) 301–312

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Simultaneous confidence bands for a percentile hyper-plane with covariates constrained in a restricted range

Sanyu Zhou [a,*], Jingjing Zhu [a], Defa Wang [b]

[a] *Institute of Finance and Economics Research, School of Urban and Regional Science, Shanghai University of Finance and Economics, 777 Guoding Road, Shanghai 200433, China*
[b] *Zhejiang College, Shanghai University of Finance and Economics, 99 South Huancheng Road, Jinhua, Zhejiang 321013, China*

ARTICLE INFO

Article history:
Received 23 June 2017
Received in revised form 27 March 2018

Keywords:
Simultaneous inference
Confidence set
Statistical simulation
Percentile hyper-plane

ABSTRACT

A simultaneous confidence band provides useful information about the plausible range of the unknown regression model. In several recent papers, confidence bands have been used for various inferential purposes; see, for example, Liu et al. (2004, 2005, 2009, 2010, 2012, 2014), Liu and Hayter (2007) [1], Liu and Lin (2009), Piegorsch et al. (2005), Han et al. (2015) and Zhou (2017). The construction of simultaneous confidence bands has a long history, going back to Working and Hotelling (1929) [2], and is often difficult, when the region over which a confidence band is required is restricted and the number of predictor variables is greater than one. In this paper we have developed methods to construct simultaneous confidence bands for a percentile hyper-plane. These methods allow us to construct the bands over arbitrary intervals, either finite or infinite; they also allow us to construct the Type I and Type II bands. In order to get an idea of the efficiency of these bands, we have also developed the confidence sets for the bands and have compared the bands under the minimum volume confidence set optimality criterion. The proposed methods are illustrated by an example.

1. Introduction

The problem addressed in this paper is that of constructing simultaneous confidence bands, both Type I and Type II, over given covariate intervals, either finite or infinite, for the percentile hyper-plane $y(\gamma, \mathbf{x}) = \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta} + z_\gamma\sigma$, where z_γ is the 100γth percentile of the standard normal distribution, for the case of the multiple linear regression model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$ with $\mathbf{e} \sim N_n(\mathbf{0}, \sigma^2\mathbf{I})$. Recently, several papers have considered various applications of the confidence bands for the mean regression hyper-plane $\widehat{y}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\widehat{\boldsymbol{\beta}}$ which is a special case of the percentile hyper-plane $\widehat{y}(\gamma, \mathbf{x}) = \mathbf{x}^{\mathrm{T}}\widehat{\boldsymbol{\beta}} + z_\gamma\widehat{\sigma}$, when $z_\gamma = 0$, that is, $\gamma = 0.5$; see, for example, [3–12].

Consider the simultaneous confidence band of the form

$$\widehat{y}(\gamma, \mathbf{x}) \in \mathbf{x}^{\mathrm{T}}\widehat{\boldsymbol{\beta}} + z_\gamma\widehat{\sigma}/\theta \pm c\widehat{\sigma}\sqrt{\mathbf{x}^{\mathrm{T}}(X^{\mathrm{T}}X)^{-1}\mathbf{x} + (z_\gamma)^2\xi} \quad \text{for every} \quad x_l \in (a_l, b_l),\ l = 1, 2, \ldots, p \tag{1}$$

where c is a critical constant that is suitably chosen so that the confidence level of the band is equal to $1 - \alpha$, and the values of θ and ξ were studied by several researchers as discussed below. Based on whether $\xi = 0$ or $\xi \neq 0$, the

* Corresponding author.
E-mail addresses: sanyu.zhou@foxmail.com (S. Zhou), weierdexiaoyao@sina.com (J. Zhu), tongji_yjs@163.com (D. Wang).

https://doi.org/10.1016/j.cam.2018.05.026
0377-0427/

band can be divided into Type I and Type II, respectively. Throughout this paper we will denote the covariate region by $\Omega = \{(x_1, \ldots, x_p)^{\mathrm{T}} : a_i \le x_i \le b_i, i = 1, 2, \ldots, p\}$, where $-\infty \le a_i < b_i \le +\infty$ $(i = 1, 2, \ldots, p)$ is given.

Without loss of generality we use the mean-centered covariates. Therefore, the ith row of the usual design matrix $\mathbf{X}$ is given by $(1, x_{i1} - \bar{x}_{\cdot 1}, x_{i2} - \bar{x}_{\cdot 2}, \ldots, x_{ip} - \bar{x}_{\cdot p})$. Let $\mathbf{X}_{(1)}$ denote the resultant of deleting the first column from $\mathbf{X}$. It is well-known that the ordinary least square estimators of $\boldsymbol{\beta}$ and σ^2 are given by $\widehat{\boldsymbol{\beta}} = (\widehat{\beta}_0, \widehat{\boldsymbol{\beta}}_{(1)}^{\mathrm{T}})^{\mathrm{T}} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{Y} = (\bar{y}, \mathbf{Y}^{\mathrm{T}}\mathbf{X}_{(1)}(\mathbf{X}_{(1)}^{\mathrm{T}}\mathbf{X}_{(1)})^{-1})^{\mathrm{T}}$ and $\widehat{\sigma}^2 = (\mathbf{Y}^{\mathrm{T}}\mathbf{Y} - \widehat{\boldsymbol{\beta}}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{Y})/\nu$, where $\nu = n - p - 1$ and

$$(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1} = \begin{pmatrix} 1/n & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & (\mathbf{X}_{(1)}^{\mathrm{T}}\mathbf{X}_{(1)})^{-1} \end{pmatrix}.$$

Let $\mathbf{P}$ be the unique positive definite square-root matrix of $(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}$, and therefore

$$\mathbf{P} = \begin{pmatrix} 1/\sqrt{n} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & (\mathbf{X}_{(1)}^{\mathrm{T}}\mathbf{X}_{(1)})^{-1/2} \end{pmatrix}.$$

It is well known that $\widehat{\boldsymbol{\beta}}$ and $\widehat{\sigma}^2$ are independent random variables with $\widehat{\boldsymbol{\beta}} \sim N_{p+1}(\boldsymbol{\beta}, \sigma^2(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1})$ and $U^2 = \widehat{\sigma}^2/\sigma^2 \sim \chi_\nu^2/\nu$. Obviously, $\mathbf{N} = \mathbf{P}^{-1}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})/\sigma \sim N_{p+1}(\mathbf{0}, \mathbf{I})$. The random variable $\mathbf{N}^{\mathrm{T}}\mathbf{N}$ has a central χ^2 distribution with $p + 1$ degrees of freedom. For convenience of notation, we decompose $\mathbf{N}$ into $\mathbf{N} = (N_1, \mathbf{N}_{(1)}^{\mathrm{T}})^{\mathrm{T}}$ where N_1 is the first element of $\mathbf{N}$, and $\mathbf{N}_{(1)}$ is resultant from deleting the first element of $\mathbf{N}$. Define two random variables $\mathbf{D}$ and $\mathbf{G}$ as follows:

$$\mathbf{D} = \begin{pmatrix} N_1 + (\sqrt{U}/\theta - 1)\sqrt{n}z_\gamma \\ \mathbf{N}_{(1)} \end{pmatrix} \text{ for } \xi = 0$$

and

$$\mathbf{G} = \begin{pmatrix} \mathbf{N} \\ (\sqrt{U}/\theta - 1)/\sqrt{\xi} \end{pmatrix} \text{ for } \xi \neq 0.$$

The conditional distribution of $\mathbf{D}$, given $U = u$, is multivariate normal, viz.

$$(\mathbf{D}|U = u) \sim N_{p+1}\left[\begin{pmatrix} (\sqrt{u}/\theta - 1)\sqrt{n}z_\gamma \\ \mathbf{0} \end{pmatrix}, \mathbf{I}\right].$$

This implies that when $z_\gamma \neq 0$, $(\mathbf{D}^{\mathrm{T}}\mathbf{D})|U = u$ is a noncentral χ^2 distribution with $p + 1$ degrees of freedom and the noncentrality parameter $\lambda = nz_\gamma^2(\sqrt{U}/\theta - 1)^2$; when $z_\gamma = 0$, $\mathbf{D}^{\mathrm{T}}\mathbf{D}$ has a central χ^2 distribution with $p + 1$ degrees of freedom. The random variable $\mathbf{G}^{\mathrm{T}}\mathbf{G}$ is the sum of two dependent variables: $(\sqrt{U}/\theta - 1)^2/\xi$ and $\mathbf{N}^{\mathrm{T}}\mathbf{N}$.

Define $\mathbf{T} = \mathbf{P}^{-1}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})/\widehat{\sigma}$. It follows from the definition of $\mathbf{T}$ that $\mathbf{T} = \mathbf{P}^{-1}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})/\widehat{\sigma} = \mathbf{N}/(\widehat{\sigma}/\sigma) = \mathbf{N}/U$ is a standard $(p + 1)$-dimensional t random vector with ν degrees of freedom, and that $\|\mathbf{T}\| \sim \sqrt{(p + 1)F_{p+1,\nu}}$ where $F_{p+1,\nu}$ denotes an F random variable with $p + 1$ and ν degrees of freedom.

Constructing the simultaneous confidence band for a percentile line has a long history. Published work has focused on the confidence band only in the simple linear regression case, that is, $p = 1$. Steinhorst and Bowden [13] constructed two types of simultaneous confidence bands over the infinite interval $(-\infty, +\infty)$; Based on Bowden [14] and using $\xi = 0, \theta = 1$, and the distribution of $\mathbf{D}^{\mathrm{T}}\mathbf{D}$, they construct a Type I band, whereas they constructed a Type II band by using $\xi = 1, \theta = 1$, and the distribution of $\mathbf{G}^{\mathrm{T}}\mathbf{G}$. On the basis of the Cauchy–Schwarz inequality, Turner and Bowden [15] generalized the procedure of Steinhorst and Bowden over the infinite interval $(-\infty, +\infty)$ by using the values of several pairs of (ξ, θ): $(0, \sqrt{\frac{2}{\nu}}\frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)})$, $(0, \sqrt{\frac{2}{\nu}\frac{\Gamma(\nu/2)}{\Gamma((\nu-1)/2)}})$, $(0, \frac{1}{\sqrt{n}})$, and $(0, \infty)$. Using $\xi = \frac{1}{2\nu}$ and $\theta = 1 - \frac{1}{4\nu}$, Thomas and Thomas [16] used the Cauchy–Schwarz inequality and the conditional distributions of $\mathbf{G}^{\mathrm{T}}\mathbf{G}$, given $U = u$, to construct symmetric Type II bands over the infinite interval $(-\infty, +\infty)$. Using $\theta = \sqrt{\frac{2}{\nu}}\Gamma(\frac{\nu+1}{2})/\Gamma(\frac{\nu}{2})$ and $\xi = \frac{\nu}{2}(\Gamma(\frac{\nu}{2})/\Gamma(\frac{\nu+1}{2}))^2 - 1$, Han et al. [17] proposed an asymmetric confidence band over the interval (a, b) by applying a simulation-based method.

Note that when $\theta = \sqrt{\frac{2}{\nu}}\Gamma(\frac{\nu+1}{2})/\Gamma(\frac{\nu}{2})$ and $\xi = \frac{\nu}{2}(\Gamma(\frac{\nu}{2})/\Gamma(\frac{\nu+1}{2}))^2 - 1$, $E(\mathbf{x}^T\widehat{\boldsymbol{\beta}} + z_\gamma\widehat{\sigma}/\theta) = \mathbf{x}^T\boldsymbol{\beta} + z_\gamma\sigma$ and $\mathrm{var}(\mathbf{x}^T\widehat{\boldsymbol{\beta}} + z_\gamma\widehat{\sigma}/\theta) = [\mathbf{x}^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{x} + (z_\gamma)^2\xi]\sigma^2$. In this paper we choose $\theta = \sqrt{\frac{2}{\nu}}\Gamma(\frac{\nu+1}{2})/\Gamma(\frac{\nu}{2})$ and $\xi = 0$ to construct the Type I band, and choose $\theta = \sqrt{\frac{2}{\nu}}\Gamma(\frac{\nu+1}{2})/\Gamma(\frac{\nu}{2})$ and $\xi = \frac{\nu}{2}(\Gamma(\frac{\nu}{2})/\Gamma(\frac{\nu+1}{2}))^2 - 1$ to construct the Type II band, respectively.

This paper is divided as follows. In Section 2 we construct the Type I and Type II bands. In Section 3 the confidence sets for the bands are developed. Section 4 presents two examples to illustrate the recommended methods. Section 5 offers some concluding remarks.