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Condition numbers for a linear function of the solution of the linear least squares problem with equality constraints

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ABSTRACT

In this paper, we consider the normwise, mixed and componentwise condition numbers for a linear function $\boldsymbol{L}x$ of the solution x to the linear least squares problem with equality constraints (LSE). The explicit expressions of the normwise, mixed and componentwise condition numbers are derived. Also, we revisit some previous results on the condition numbers of linear least squares problem (LS) and LSE. It is shown that some previous explicit condition number expressions on LS and LSE can be recovered from our new derived condition numbers' formulas. The sharp upper bounds for the derived normwise, mixed and componentwise condition numbers are obtained, which can be estimated efficiently by means of the classical Hager–Higham algorithm for estimating matrix one-norm. Moreover, the proposed condition estimation methods can be incorporated into the generalized QR factorization method for solving LSE. The numerical examples show that when the coefficient matrices of LSE are sparse and badly-scaled, the mixed and componentwise condition numbers can give sharp perturbation bounds, on the other hand normwise condition numbers can severely overestimate the exact relative errors because normwise condition numbers ignore the data sparsity and scaling. However, from the numerical experiments for random LSE problems, if the data is not either sparse or badly scaled, it is more suitable to adopt the normwise condition number to measure the conditioning of LSE since the explicit formula of the normwise condition number is more compact.

1. Introduction

The least squares problem with equality constraints (LSE) has the following form:

$$\text{LSE}: \qquad \min_{x\in\mathbb{R}^n} \|Ax-b\|_2 \text{ subject to } Cx = d, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{p\times n}$, $b \in \mathbb{R}^m$, $d \in \mathbb{R}^p$ and $m+p \ge n \ge p$. The rank conditions [1]

$$\operatorname{rank}(C) = p \text{ and } \operatorname{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) = n \tag{1.2}$$

guarantee the existence of the unique solution of LSE [1,2]

$$x = \mathcal{K}b + C_A^{\dagger}d,$$

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where

$$\mathcal{K} = (A\mathcal{P})^{\dagger}, \quad \mathcal{P} = I_n - C^{\dagger}C, \quad C_A^{\dagger} = (I_n - \mathcal{K}A)\, C^{\dagger}, \tag{1.3}$$

and $B^{\dagger}$ is the Moore–Penrose inverse of B [1]. Under the rank condition $\mathrm{rank}(C) = p$ the equality constraints $Cx = d$ in (1.1) are consistent, thus LSE (1.1) has solutions. The second rank condition of (1.2) guarantees the uniqueness of the solution to (1.1). On the other hand, the augmented system also defines the unique solution x as follows:

$$\mathcal{A}\mathbf{x} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & C \\ \mathbf{0} & I_m & A \\ C^{\top} & A^{\top} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \lambda \\ r \\ x \end{bmatrix} = \begin{bmatrix} d \\ b \\ \mathbf{0} \end{bmatrix} := \mathbf{b}, \tag{1.4}$$

where $A^{\top}$ is the transpose of A, I_m denotes the $m \times m$ identity matrix, $\mathbf{0}$ is the zeros matrix with conformal dimension, $\lambda \in \mathbb{R}^p$ is a vector of Lagrange multipliers, and r is the residual vector $r = b - Ax$. As stated in [2,3], when the rank condition (1.2) is satisfied, $\mathcal{A}$ is nonsingular and its inverse has the following expression:

$$\mathcal{A}^{-1} = \begin{bmatrix} (AC_A^{\dagger})^{\top}AC_A^{\dagger} & -(AC_A^{\dagger})^{\top} & (C_A^{\dagger})^{\top} \\ -AC_A^{\dagger} & I_m - (A\mathcal{P})\mathcal{K} & \mathcal{K}^{\top} \\ C_A^{\dagger} & \mathcal{K} & -\left((A\mathcal{P})^{\top}(A\mathcal{P})\right)^{\dagger} \end{bmatrix}. \tag{1.5}$$

When $C = \mathbf{0}$ and $d = \mathbf{0}$, LSE is reduced to the classical linear least squares problem (LS) as follows

$$\mathrm{LS}: \quad \min_{x \in \mathbb{R}^n} \|Ax - b\|_2. \tag{1.6}$$

In this case, we know that the rank condition (1.2) becomes $\mathrm{rank}(A) = n$. Thus LS has the unique LS solution $x = A^{\dagger}b = (A^{\top}A)^{-1}A^{\top}b$.

The LSE problem has many applications such as in the analysis of large scale structures [4], and the solution of the inequality constrained least square problem [5] etc. The algorithms and perturbation analysis of LSE can be found in several papers [1–9] and references therein.

Perturbation theory is important in matrix computation, since they can give error bounds for the computed solution. Especially, condition number measures the *worst-case* sensitivity of an input data with respect to *small* perturbations on it; see the recent monograph [10] and references therein. Rice in [11] gave a general theory of condition numbers. Let $\psi : \mathbb{R}^p \to \mathbb{R}^q$ be a mapping, where $\mathbb{R}^p$ and $\mathbb{R}^q$ are the usual p- and q-dimensional Euclidean spaces equipped with some norms, respectively. If ψ is continuous and Fréchet differentiable in the neighborhood of $u_0 \in \mathbb{R}^p$ then, according to [11], the *relative normwise condition number* of ψ at u_0 is given by

$$\mathrm{cond}^{\psi}(u_0) := \lim_{\varepsilon \to 0} \sup_{\|\Delta u\| \le \varepsilon} \left(\frac{\|\psi(u_0 + \Delta u) - \psi(u_0)\|}{\|\psi(u_0)\|} \Big/ \frac{\|\Delta u\|}{\|u_0\|} \right) = \frac{\|\mathrm{d}\psi(u_0)\| \, \|u_0\|}{\|\psi(u_0)\|}, \tag{1.7}$$

where $\mathrm{d}\psi(u_0)$ is the Fréchet derivative of ψ at u_0. Condition number can tell us the loss of the precision in finite precision computation of a problem. With the backward error of a problem, the relative error of the computed solution can be bounded by the product of condition number and backward error.

When the data is sparse or badly-scaled, componentwise perturbation analysis [12,13] has been proposed to investigate the mixed condition numbers and componentwise condition numbers [14] of the problem in matrix computation. The mixed condition numbers use the componentwise error analysis for the input data, while the normwise error analysis for the output data. On the other hand, the componentwise condition numbers use the componentwise error analysis for both input and output data. Consequently, the perturbation bounds based on the mixed and componentwise condition numbers are more effective and sharper than those based on the normwise condition number when the data is sparse or badly scaled because the normwise condition number defined in (1.7) does not take account of the structure of both input and output data with respect to scaling and/or sparsity.

In some situations, the conditionings of particular components of a solution are different. Thus it is suitable to consider the condition numbers of a linear function of the solution. These type condition numbers had been studied for the LS problem [15,16], the weighted LS problem [17], the total least squares problems [18,19], the indefinite LS problem [20] and the LSE problem [21], etc. In this paper, we will investigate the sensitivity of a linear function $\boldsymbol{L}x$ of the LSE solution x with respect to perturbations on the data A, C, b and d. First, let us introduce the following mapping

$$\begin{aligned} &\Phi : \mathbb{R}^{mn} \times \mathbb{R}^{pn} \times \mathbb{R}^{m} \times \mathbb{R}^{p} \to \mathbb{R}^{k} \\ &\Phi(\mathrm{vec}(A),\ \mathrm{vec}(C),\ b,\ d) := \boldsymbol{L}\left(\mathcal{K}b + C_A^{\dagger}d\right), \end{aligned} \tag{1.8}$$

where $\mathrm{vec}(A)$ is a vector obtained by stacking the columns of a matrix A one by one (see [22] for details), and $\boldsymbol{L}$ is an k-by-n, $k \le n$, matrix introduced for the selection of the solution components. For example, when $\boldsymbol{L} = I_n$ ($k = n$), all the n components of the solution x are equally selected. When $\boldsymbol{L} = e_i^{\top}$ ($k = 1$), the ith row of I_n, then only the ith component of the solution is selected. The matrix $\boldsymbol{L}$ is not perturbed in the text.