Journal of Computational and Applied Mathematics 343 (2018) 206–216

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

High-order skew-symmetric differentiation matrix on symmetric grid☆

Kai Liu[a], Wei Shi[b,*]

[a] College of Applied Mathematics, Nanjing University of Finance& Economics, Nanjing 210023, PR China
[b] School of Physical and Mathematical Sciences, Nanjing Tech University, Nanjing 211816, PR China

ARTICLE INFO

Article history:
Received 15 December 2017
Received in revised form 17 April 2018

Keywords:
Numerical stability
Partial differential equations
Finite difference methods
Skew-symmetric differentiation matrices
Order conditions

ABSTRACT

Hairer and Iserles (2016) presented a detailed study of skew-symmetric matrix approximation to a first derivative which is proved to be fundamental in ensuring stability of discretisation for evolutional partial differential equations with variable coefficients. An open problem is proposed in that paper which concerns about the existence and construction of the perturbed grid that supports high-order skew-symmetric differentiation matrix for a given grid and only the case $p = 2$ for this problem have been solved. This paper is an attempt to solve the problem for any $p \geqslant 3$. We focus ourselves on the symmetric grid and prove the existence of the perturbed grid for arbitrarily high order p and give in detail the construction of the perturbed grid. Numerical experiments are carried out to illustrate our theory.

1. Introduction

Stability is a necessary condition to ensure convergence of a numerical method for partial differential equations (PDEs) of evolution. Moreover, in practice, the efficiency of the numerical method is closely related to its stability behaviour. The analysis of numerical stability is normally based on the PDE

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f, \qquad x \in \Omega, \qquad t > 0 \tag{1}$$

with suitably smooth initial and boundary conditions. Here, $u = u(x, t)$, $\Omega \subseteq \mathbb{R}^d$, $\mathcal{L}$ is a linear, time-independent differential operator. Consider the discretisation of the PDE (1) with the form

$$\mathbf{u}_N^{n+1} = \mathcal{A}_N \mathbf{u}_N^n + \mathbf{f}_N^n, \tag{2}$$

where $\mathbf{u}_N^n = (u_{N,1}^n, u_{N,2}^n, \ldots, u_{N,N}^n)$ with $u_{N,m}^n \approx u(x_m, n\Delta t)$, x_m the grid point and $\Delta t > 0$ the time step. $\mathcal{A}_N$ is a finite dimensional approximation of L on some grid. The *stability* of (2) means the uniform well-posedness of the operator $\{\mathcal{A}_N\}$ as

☆ The research was supported in part by the Natural Science Foundation of China under Grants 11501288, 11701271 and 11671200, by the Specialized Research Foundation for the Doctoral Program of Higher Education under Grant 20100091110033, by the 985 Project at Nanjing University under Grant 9112020301, by A Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions, by the Natural Science Foundation of Jiangsu Province under Grant BK20150934, and by the Natural Science Foundation of the Jiangsu Higher Education Institutions under Grants 16KJB110010 and 14KJB110009.

* Corresponding author.
E-mail address: shuier628@163.com (W. Shi).

https://doi.org/10.1016/j.cam.2018.04.038
0377-0427/

$N \to \infty$ in the time interval $[0, T]$ (see [1]), which together with *consistency* of (2) guarantees *convergence*, i.e., $\mathbf{u}_N^n$ converges pointwise to the exact solution of (1).

If the coefficients of $\mathcal{L}$ are constant, several powerful techniques can be used to investigate the numerical stability of discretisation of (1), such as Fourier transformation and the corresponding Von Neumann condition for analysing stability, eigenvalue analysis and energy methods. For topics on this subject, we refer the reader to [1–7] and the references therein.

Once the coefficients of $\mathcal{L}$ depend on spatial variable x, the analysis becomes considerably complicated. A natural idea is to freeze the coefficients and check the stability using above-mentioned methods. However, as it turns out, the stability of the method for frozen problem is neither sufficient nor necessary for stability in the variable coefficient case. In [8], E. Hairer and A. Iserles showed that the skew-symmetric differentiation matrix approximation for first derivative is fundamental in ensuring stability of discretisation for evolutional PDEs with variable coefficients (see [8], Theorem 1). A wide range of PDEs such as Liouville equation, Convection–diffusion equation, the Fokker–Planck equation, can be discretized stably once first space derivative are approximated by skew-symmetric matrices. However, it has been proved in [9] that the highest order a skew-symmetric differentiation matrix could get on a uniform grid is just two. With mild perturbation of the uniform grid, the third and fourth-order differentiation matrices for first derivative are constructed on the modified grid. Later, this result is extended to a more general case [8]: given arbitrary grid, not necessarily uniform, the existence of a perturbed grid which supports skew-symmetric differentiation matrices of a given order p is discussed. However, a constructive proof is presented only for $p = 2$ based on a necessary condition given in [8] leaving $p \geqslant 3$ as an open problem. In [10], E. Hairer and A. Iserles derived banded and skew-symmetric differentiation matrices of orders up to 6 based on a perturbation of a uniform grid. In this paper, we will addresses the open problem for general symmetric grid, i.e., given arbitrary symmetric grid, we will discuss in detail the existence and construction of symmetric perturbed grid that supports skew-symmetric differentiation matrix approximation of arbitrary order $p \geqslant 3$ for first derivative.

The outline of this paper is as follows. In Section 2, we introduce some notation and revisit some existing results from [8]. In Section 3, we commence from $p = 3$ and prove the existence of symmetric perturbed grid that supports skew-symmetric differentiation matrix approximation of order 3 for first derivative. Then the analysis is extended to the case of arbitrary order p. An efficient approach to construct the perturbed grid is also provided. Numerical experiments are carried out in Section 4. The last section is devoted to conclusions.

2. Notation and some existing results

Following [8], we assume for simplicity zero Dirichlet boundary conditions and consider the one-dimensional case. Let $\Omega = [0, 1]$ and the grid be $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$. An $N \times N$ differentiation matrix $\mathcal{D}$ defined on this grid is of order p, if

$$u'(x_m) = \sum_{k=1}^{N} \mathcal{D}_{m,k} u(x_k), \quad m = 1, 2, \ldots, N,$$

for every polynomial $u(x)$ of degree p that vanishes at the endpoints.

Theorem 2.1 (*[8]*). *Consider a differentiation matrix $\mathcal{D}$ of order p ($p \geqslant 2$) corresponding to the grid be $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$. The necessary condition for $\mathcal{D}$ to be skew-symmetric is*

$$R_N^{[1]} = R_N^{[2]} = \cdots = R_N^{[2p-3]} = 0 \tag{3}$$

with

$$R_m^{[s]} = \sum_{k=1}^{m} x_k^s (1 - x_k)[(s+1) - (s+3)x_k], \quad s \in \mathbb{N}. \tag{4}$$

Let $q \geqslant 1$ and $p = q + 1$. The order conditions (3) are sufficient for the existence of a skew-symmetric differentiation matrix of order p with bandwidth $2q + 1$.

If the grid is symmetric, i.e., $x_m = 1 - x_{N+1-m}$ for $m = 1, \ldots, N$, then $R_N^{[1]} = R_N^{[3]} = \cdots = R_N^{[2p-3]} = 0$ automatically, therefore the order conditions (3) reduce to

$$R_N^{[2]} = R_N^{[4]} = \cdots = R_N^{[2p-4]} = 0. \tag{5}$$

In practice, the nature of grid is determined by the coefficients of $\mathcal{L}$. That means we cannot expect order conditions (3) (or (5) if the grid is symmetric) hold in the first place. Our concern is that given a grid $\{x_m\}_{m=0}^{N+1}$ to find a perturbed grid $\{\tilde{x}_m\}_{m=0}^{N+1}$ as close to the given grid as possible (e.g., $\tilde{x}_m = x_m + O(N^{-\alpha})$ for $m = 1, \ldots, N$ and $\alpha \geqslant 1$) such that the order conditions (3) hold. Suppose that the grid $\{x_m\}_{m=0}^{N+1}$ can be expressed as the form $x_m = g(m/(N+1))$, $m = 0, 1, \ldots, N+1$, where g is a strictly monotonically increasing, sufficiently smooth function that maps [0, 1] to itself, the following lemma gives a necessary condition for the existence of the perturbed grid.