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A new family of methods for single and multiple roots

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ABSTRACT

We present a new family of iterative methods for multiple and single roots of nonlinear equations. This family contains as a special case the authors' family for finding simple roots from Herceg and Herceg (2015). Some well-known classical methods for simple roots, for example Newton, Potra–Pták, Newton–Steffensen, King and Ostrowski's methods, belong to this family, which implies that our new family contains modifications of these methods suitable for finding multiple roots. Convergence analysis shows that our family contains methods of convergence order from 2 to 4. The new methods require two function evaluations and one evaluation of the first derivative per iteration, so all our fourth order methods are optimal in terms of the Kung and Traub conjecture. Several examples are presented and compared. Through various test equations, relevant numerical experiments strongly support the claimed theory in this paper. Extraneous fixed points of the iterative maps associated with the proposed methods are also investigated. Their dynamics is explored along with illustrated basins of attraction for various polynomials.

1. Introduction

We consider a family of iterative methods for finding a multiple root α of multiplicity m of the nonlinear equation $f(x) = 0$, i.e. $f^{(k)}(\alpha) = 0$, $k = 0, 1, \ldots, m-1$ and $f^{(m)}(\alpha) \neq 0$. We assume that f has sufficient number of continuous derivatives in a neighborhood of α.

Cauchy's method, [1], is defined by (1), (2) with

$$x_{n+1} = F(x_n), \quad n = 0, 1, \ldots \tag{1}$$

where

$$F(x) = x - u(x)\,\phi(x) \tag{2}$$

$$u(x) = \frac{f(x)}{f'(x)}$$

$$\phi(x) = \frac{2}{1 + \sqrt{1 - 2L(x)}}$$

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Table 1
Padé approximations for $p, q = 0, 1, 2, 3$.

p	q			
	0	1	2	3
0	1	$\frac{-2}{z-2}$	$\frac{-4}{z^2+2z-4}$	$\frac{-4}{z^3+z^2+2z-4}$
1	$\frac{z}{2}+1$	$\frac{z-2}{2(z-1)}$	$\frac{4-4z}{z^2-6z+4}$	$\frac{16-20z}{z^3+6z^2-28z+16}$
2	$\frac{z^2}{2}+\frac{z}{2}+1$	$\frac{z^2+6z-8}{10z-8}$	$\frac{z^2-6z+4}{3z^2-8z+4}$	$\frac{-2\left(3z^2-8z+4\right)}{z^3-12z^2+20z-8}$
3	$\frac{5z^3}{8}+\frac{z^2}{2}+\frac{z}{2}+1$	$\frac{3z^3+8z^2+36z-40}{8(7z-5)}$	$\frac{z^3+12z^2-44z+24}{28z^2-56z+24}$	$\frac{z^3-12z^2+20z-8}{4\left(z^3-5z^2+6z-2\right)}$

and

$$L(x) = \frac{f''(x) f(x)}{f'(x)^2}. \tag{3}$$

This is a well-known third-order method. There are numerous modifications of this method which are of the order four, for example, in the papers [2,3] its order is increased from three to four without the additional computational costs. However, in many cases, it is expensive to compute second derivatives and their practical applications are restricted rigorously, so that in past, Newton's method was frequently used to solve such nonlinear equations because of its higher computational efficiency. Recently, many modified methods, free from the second derivative, have been studied in [3,4] and the literature cited therein. These methods have the order of convergence four, and therefore may be very interesting. The second derivative in (3) can be omitted by evaluating the function in different points.

In [5] we considered simple modifications of Cauchy's method and we obtained three methods of fourth order. Two of these methods are free of the second derivative.

Using Padé approximation of order (k, m) to function $2/\left(1+\sqrt{1-2z}\right)$ at 0, we obtained in [5] an infinite family of iterative methods for finding single roots. Convergence analysis shows that this family contains methods of convergence order from 2 to 4.

In this paper our optimal family of methods for finding single roots is generalized, in a simple way, to become a new optimal family of methods for multiple roots. Also, this family contains methods of convergence order from 2 to 4. Since some well-known classical methods for simple roots, such as Newton, Potra–Pták, Newton–Steffensen, King and Ostrowski's methods, belong to our family from [5], these methods are also generalized and become suitable for finding multiple roots. Our new family for multiple roots can be used for single and multiple roots alike, i.e. $m = 1, 2, \ldots$.

To provide an overview of convergence of the considered family of methods, basins of attraction and extraneous fixed points are observed. For the considered examples, it is shown that the basins of attraction are the same for each m. Furthermore, the extraneous fixed points do not depend on m. Also, our family of methods contains a subfamily which does not have extraneous fixed points. None of the methods from [6–15] have these properties.

From now on we consider a family of iterative methods

$$x_{n+1} = F_{p,q}(x_n), \quad n = 0, 1, \ldots \tag{4}$$

with the iteration function $F_{p,q}$ of the form

$$F_{p,q}(x) = x - m\frac{f(x)}{f'(x)}\varphi_{p,q}(\sigma_m(x)), \quad p, q = 0, 1, \ldots, \tag{5}$$

where the function $\varphi_{p,q}$ is defined as a Padé approximation of order (p, q) to function $2/\left(1+\sqrt{1-2z}\right)$ at 0 and

$$\sigma_m(x) = 2\sqrt[m]{\frac{f\left(x - m\frac{f(x)}{f'(x)}\right)}{f(x)}}. \tag{6}$$

We can organize the considered Padé approximations into a table. Table 1 shows Padé approximations for $p, q = 0, 1, 2, 3$.

A variant of Newton's method for obtaining multiple roots, given in [16], is quadratically convergent, and given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots. \tag{7}$$

This well-known iterative method for computing approximations of α can be viewed as the method (4) and (5), with $\varphi_{0,0}(x) = 1$.

In order to improve the convergence of iterative methods for multiple roots, some researchers, such as Liu et al. [6], Neta et al. [7,8], Chun et al. [9,10,17], Sharma et al. [11], Thukral [12], Zhou et al. [14,15] have developed some iterative methods with a higher order of convergence. Some of these methods are of order three [10,15,17,18], while others are of order four [6,7,11,12,14,15]. Liu et al. [19] have 6 families of fourth order.