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# A stabilized normal form algorithm for generic systems of polynomial equations

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ABSTRACT

We propose a numerical linear algebra based method to find the multiplication operators of the quotient ring $\mathbb{C}[x]/I$ associated to a zero-dimensional ideal $I$ generated by $n$ $\mathbb{C}$-polynomials in $n$ variables. We assume that the polynomials are generic in the sense that the number of solutions in $\mathbb{C}^n$ equals the Bézout number. The main contribution of this paper is an automated choice of basis for $\mathbb{C}[x]/I$, which is crucial for the feasibility of normal form methods in finite precision arithmetic. This choice is based on numerical linear algebra techniques and it depends on the given generators of $I$.



## 1. Introduction

Consider the following problem. Given $n$ polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ with $k$ an algebraically closed field, find all the points $x \in k^n$ where they all vanish: $f_1(x) = \cdots = f_n(x) = 0$. Here, we will work over the complex numbers $k = \mathbb{C}$. The ring of all polynomials in the $n$ variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{C}$ is denoted by $\mathbb{C}[x_1, \ldots, x_n]$. For short, we will denote $x = (x_1, \ldots, x_n)$ and an element $f \in \mathbb{C}[x]$ can be written as

$$f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha$$

where we used the short notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The *support* $S(f)$ of $f$ is defined as

$$S(f) = \{\alpha \in \mathbb{Z}^n_{\geq 0} : c_\alpha \neq 0\}.$$

A set of $n$ polynomials $\{f_1, \ldots, f_n\} \subset \mathbb{C}[x]$ defines a *square ideal*

$$I = \langle f_1, \ldots, f_n \rangle = \{g_1 f_1 + \cdots + g_n f_n : g_1, \ldots, g_n \in \mathbb{C}[x]\} \subset \mathbb{C}[x].$$

The *affine variety* associated to $I$ is

$$\mathbb{V}(I) = \{x \in \mathbb{C}^n : f(x) = 0, \forall f \in I\} = \{x \in \mathbb{C}^n : f_1(x) = \cdots = f_n(x) = 0\}.$$

In this paper, we assume that the variety $\mathbb{V}(I)$ consists of finitely many points $\{z_1, \ldots, z_N\} \subset \mathbb{C}^n$. Such a variety is called *0-dimensional.*

A well known result in algebraic geometry states that the quotient ring $k[x_1, \ldots, x_n]/I$ with $I \subset k[x_1, \ldots, x_n]$ a 0-dimensional ideal and $k$ an algebraically closed field is isomorphic as a $k$-algebra to a finite dimensional $k$-vectorspace $V$ with multiplication defined by a pairwise commuting set of $n$ square matrices over $k$. This set of matrices corresponds

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to a set of generators of $k[x_1, \ldots, x_n]/I$ and the size of each matrix is equal to the number of points in $\mathbb{V}(I) \subset k^n$, counting multiplicities. Once the (generating) multiplication matrices are known in some basis, we can answer several questions about the variety $\mathbb{V}(I)$. For example, we can retrieve the solutions of the system by computing their eigenstructure and we can evaluate any polynomial on $\mathbb{V}(I)$. Our goal is to compute the multiplication matrices in a numerically stable way for square ideals satisfying some genericity assumptions.

There are many approaches to the problem of solving systems of polynomial equations. The different methods are often subdivided in homotopy methods, subdivision methods and algebraic methods. Homotopy continuation uses Newton iteration to track solution paths, starting from a simple initial system and gradually transforming it into the target system. These ideas have led to highly successful solvers [1,2]. However, performing some numerical experiments one observes that for large systems some solutions might be lost along the way. The continuation gives up on certain paths when, for example, they seem to be diverging to infinity or they enter an ill-conditioned region. Normal form algorithms belong to the category of algebraic methods. The earliest versions of these algorithms use Groebner bases [3,4] and doing so they make an implicit choice of basis for $\mathbb{C}[x]/I$. It turns out that these methods are numerically unstable and infeasible for large systems of equations (high degree, many variables). More recent algorithms are based on *border bases* [5–7]. Essentially, they fix a basis $\mathcal{O}$ for $\mathbb{C}[x]/I$ and construct the multiplication matrices of the coordinate functions by calculating the normal forms of $x_1 \cdot \mathcal{O}, \ldots, x_n \cdot \mathcal{O}$ with respect to $\mathcal{O}$. Border bases are a generalization of Groebner bases and they can be used to enhance the numerical stability of normal form algorithms. However, there are no algorithms that make a choice of $\mathcal{O}$ based on the conditioning of the normal form computation problem. This is mentioned as an open problem in [7]. In this paper we present such an algorithm for generic systems that makes an automatic choice of $\mathcal{O}$, which does not necessarily correspond to a Groebner basis, nor to a border basis. What is meant by 'generic systems' is explained in Section 2. The goal is to cover the generic, dense case to illustrate the effectiveness of the idea. The connection with resultant algorithms for dense systems is established. This suggests that the techniques can be generalized to sparse systems of equations. Such a generalization will follow from the sparse variant of the Macaulay resultant algorithm, see for instance [8].

In the following section we discuss our genericity assumptions and some properties of the systems that satisfy them. Section 3 briefly reviews the multiplication maps in $\mathbb{C}[x]/I$ and their properties. We give a short motivation in Section 4 by discussing some aspects of Macaulay's resultant construction and border bases algorithms that are generalized in our approach. In Section 5 we introduce *Macaulay matrices*. The Macaulay matrices as defined in [9,10] are referred to as dense Macaulay matrices here. Section 6 presents the algorithm and some connections with border bases and Macaulay resultants. In the final section we present some numerical experiments.

## 2. Generic total degree systems

We say that a polynomial $f \in \mathbb{C}[x] \setminus \{0\}$ is of degree $d$ if

$$\max_{\alpha \in S(f)} |\alpha| = d,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We denote $\deg(f) = d$. Accordingly, we say that a square polynomial system in $n$ variables given by $\{f_1, \ldots, f_n\}$ is of degree $(d_1, \ldots, d_n)$ if $\deg(f_i) = d_i, i = 1, \ldots, n$. A polynomial $f \in \mathbb{C}[x] \setminus \{0\}$ is called *homogeneous* of degree $d$ if $|\alpha| = d, \forall \alpha \in S(f)$.

Consider the *projective n-space*

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim,$$

where $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ iff $a_i = \lambda b_i, i = 0, \ldots, n, \lambda \in \mathbb{C} \setminus \{0\}$. We can interpret $\mathbb{P}^n$ as the union of $n+1$ copies of $\mathbb{C}^n$, each of them given by putting one of the coordinates equal to 1. We will also think of $\mathbb{P}^n$ as the union of $\mathbb{C}^n$ corresponding to $x_0 = 1$ and the set $\{x_0 = 0\}$, called the *hyperplane at infinity*. For more on projective space, see [3]. Note that the equation $f = 0$ with $f \in \mathbb{C}[x_0, \ldots, x_n]$ is well defined over $\mathbb{P}^n$ if and only if $f$ is homogeneous. Starting from a polynomial $f \in \mathbb{C}[x]$ in $n$ variables of degree $d$, we can obtain a homogeneous polynomial $f^h \in \mathbb{C}[x_0, \ldots, x_n]$, called the *homogenization* of $f$ as

$$f^h = x_0^d f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

The following theorem was proved by Étienne Bézout for the intersection of algebraic plane curves in $\mathbb{P}^2$. The generalization is often referred to as Bézout's theorem.

**Theorem 1** (*Bézout*). *A system of $n$ homogeneous equations of degree $(d_1, \ldots, d_n)$ in $n+1$ variables with a finite number of solutions in $\mathbb{P}^n$ has exactly $d_1 \cdots d_n$ solutions in $\mathbb{P}^n$, counting multiplicities.*

**Proof.** The theorem is a corollary of Theorem 7.7 in [11]. □

It is not difficult to show that for *almost all* systems with degree $(d_1, \ldots, d_n)$, all $d_1 \cdots d_n$ solutions lie in the overlapping part of the affine charts of $\mathbb{P}^n$ [4]. Hence, if the $n$ homogeneous equations in $n+1$ variables of Theorem 1 are the homogenizations of $n$ affine equations $f_1 = \cdots = f_n = 0$ in $n$ variables, all of the $d_1 \cdots d_n$ solutions correspond to points in $\mathbb{C}^n \subset \mathbb{P}^n$.