Journal of Computational and Applied Mathematics 342 (2018) 225–248

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Analytic techniques for option pricing under a hyperexponential Lévy model

Daniel Hackmann

Institute of Financial Mathematics and Applied Number Theory, Johannes Kepler University, Altenbergerstraße 69 4040 Linz, Austria

ARTICLE INFO

Article history:
Received 17 May 2017
Received in revised form 3 October 2017

Keywords:
Lévy process
Hyperexponential
Option pricing
Greeks
Implied volatility
Asymptotic expansion

ABSTRACT

We develop series expansions in powers of q^{-1} and $q^{-1/2}$ of solutions of the equation $\psi(z) = q$, where $\psi(z)$ is the Laplace exponent of a hyperexponential Lévy process. As a direct consequence we derive analytic expressions for the prices of European call and put options and their Greeks (Theta, Delta, and Gamma) and a full asymptotic expansion of the short-time Black–Scholes at-the-money implied volatility. Further we demonstrate how the speed of numerical algorithms for pricing exotic options, which are based on the Laplace transform, may be increased.

1. Introduction

A hyperexponential Lévy process X is one with a Lévy measure of the form

$$\nu(\mathrm{d}x) = \mathbb{I}(x < 0)\sum_{\ell=1}^{\hat{N}} \hat{a}_\ell \hat{\rho}_\ell e^{\hat{\rho}_\ell x}\mathrm{d}x + \mathbb{I}(x > 0)\sum_{\ell=1}^{N} a_\ell \rho_\ell e^{-\rho_\ell x}\mathrm{d}x,$$

where the $\{\hat{a}_\ell\}_{1\le\ell\le\hat{N}}$ and $\{a_\ell\}_{1\le\ell\le N}$ are all positive real numbers and $0 < \rho_1 < \rho_2 < \cdots < \rho_{N-1} < \rho_N$ and $0 < \hat{\rho}_1 < \hat{\rho}_2 < \cdots < \hat{\rho}_{N-1} < \hat{\rho}_N$ hold. The Laplace exponent $\psi(z) := \frac{1}{t}\log\left(\mathbb{E}\left[e^{zX_t}\right]\right)$ has the form

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + z\sum_{\ell=1}^{N}\frac{a_\ell}{\rho_\ell - z} - z\sum_{\ell=1}^{\hat{N}}\frac{\hat{a}_\ell}{\hat{\rho}_\ell + z}, \quad -\hat{\rho}_1 < \mathrm{Re}(z) < \rho_1, \tag{1.1}$$

where $a \in \mathbb{R}$ and $\sigma \ge 0$. When $\sigma > 0$ hyperexponential processes are also called hyperexponential diffusions or hyperexponential jump diffusions in the literature.

Despite their apparent simplicity – they are compound Poisson processes plus a Brownian motion component when $\sigma > 0$ – they have been studied extensively in the literature for a number of reasons. First, hyperexponential processes are dense in the $\mathcal{CM}$-class of processes, i.e. those Lévy processes with completely monotone jump densities (also known as generalized hyperexponential processes) [1]. The $\mathcal{CM}$-class includes infinite activity models like the Variance Gamma (VG) process, the Normal Inverse Gaussian (NIG) process, and the CGMY/Kobol/Generalized Tempered Stable process, which have become very popular in finance. Second, there are a number of fast and accurate algorithms that exploit this first quality, i.e. methods by which a $\mathcal{CM}$-class process can be approximated by a hyperexponential process arbitrarily well [2,3].

E-mail address: dan@danhackmann.com.
URL: http://www.danhackmann.com.

https://doi.org/10.1016/j.cam.2018.03.036

Third, because $\psi(z)$ can be extended to a rational function with real poles on $\mathbb{C}$, hyperexponential processes are "analytically tractable". For example, we have analytic expressions for the Laplace transform (in t) of the distribution of X_t (see Theorem 2) and the Wiener–Hopf factors [4]. For financial applications, under the assumption of an exponential model for the stock price, analytic expressions for the Laplace transform of the prices of barrier [1,5,6] and look-back [7] options, for the double Laplace transform of the price of an Asian option [8], and for the prices of Russian options and for perpetual American strangles [4,9] are known. If we restrict $N = \hat{N} = 1$ to get the so-called double exponential or Kou model, we have also analytic expressions for prices of European call and put options and European options on futures contracts [10], as well as perpetual American options [11].

In almost all of the cases mentioned above, the formula for the derivative price, or the Laplace transformed price, is expressed in terms of the solutions of the equation

$$\psi(z) = q, \quad q > 0. \tag{1.2}$$

If we exclude those cases where there are fewer than four solutions, then the solutions need to be determined numerically. As a practical matter, finding solutions to (1.2) is a time consuming part of the algorithm for inverting the Laplace transform to obtain option prices (Asian options, barrier options, look-back options), especially because it becomes necessary to solve (1.2) for $q \in \mathbb{C}$.

The main idea behind this article is straightforward: we develop convergent series in powers of q^{-1} (when $\sigma = 0$) and $q^{-1/2}$ (when $\sigma > 0$) of the solutions of (1.2) for $q \in \mathbb{C}$ with $|q|$ large enough. Since the series converge quite rapidly, an immediate consequence is that the (truncated) series may be used to speed up algorithms for determining derivative prices based on numerical inversion of the Laplace transform.

While this is a useful result, further, interesting results follow from the main idea. We are also able to use the expansions to develop analytic expressions for the prices European call and put options and their Greeks. This is rather rare in exponential Lévy models, to the best of the author's knowledge there are only two other Lévy processes for which this is true: (a) Merton's model [12] and (b) Kou's model [10]. The resulting expressions involve series of functions in T, the time of expiry of the option, which when $\sigma = 0$ are, in fact, just Taylor series. In the at-the-money (ATM) case, when $\sigma > 0$, the formulas are essentially series in powers of $T^{1/2}$; this allows us to develop a full asymptotic expansion of the short-time ATM Black–Scholes implied volatility. Implied volatiles, together with short-time asymptotic expansions of call option prices, have seen a large amount of recent interest in the financial mathematics literature owing to their application to the calibration problem (see for example [13] and the references therein).

It should be noted that we are generalizing Kou's results [10]. While Kou also develops analytic formulas for European call and put option prices, his approach relies on the decomposition of sums of double exponential random variables; this technique does not seem to have a natural extension to the general case, where the number of exponential factors in the Lévy density exceeds two.

Our approach is therefore rather different and analytical in nature, relying on results from complex analysis and the theory of Laplace transforms. We devote Section 2 of the article to reviewing the relevant theory and developing notation. In Section 3 we gather some key results for hyperexponential processes and develop the series expansions of the solutions of (1.2). Then in Section 4 we develop analytic formulas for European option prices and Greeks, derive a full asymptotic expansion of the ATM implied volatility, and consider several numerical examples. Throughout the paper, effectiveness and efficiency of the techniques are demonstrated with numerical examples. Software used to compute the various examples given throughout the article can be obtained from the author's website.

2. Tools from complex analysis

2.1. Basic notation

Assuming $R > 0$ and $z_0 \in \mathbb{R}$ we define

$$\mathbb{C}^+ := \{z \in \mathbb{C} : z \notin (-\infty, 0]\}, \quad \mathbb{C}_R := \{z \in \mathbb{C} : |z| > R\}, \quad \text{and} \quad \mathbb{H}_{z_0} := \{z \in \mathbb{C} : \mathrm{Re}(z) > z_0\},$$

and using these $\mathbb{C}_R^+ := \mathbb{C}^+ \cap \mathbb{C}_R$ and $\mathbb{H} := \mathbb{H}_0$. The notation $\mathbb{Z}^+$ refers to the non-negative integers, with the analogous meaning for the notation $\mathbb{Z}^-$. We will use B to denote an open ball in $\mathbb{C}$ centred at 0, and B_0 to denote a punctured open ball excluding the point 0. If we want to be specific about the radius R we will write $B(R)$ and $B_0(R)$.

The collection of solutions w of the equation $w^k = z$, $k \in \mathbb{N}$ is denoted $z_m^{1/k}$. It follows that $z_m^{1/k}$ is a multi-valued function (see pg. 24 in [14] for a rigorous definition) taking exactly k values for all $z \neq 0$. The principal branch of $z_m^{1/k}$ will be denoted simply $z^{1/k}$. As usual, the principal branch is that branch for which $1^{1/k} = 1$. Further, we define $z_m^{n/k} := (z_m^{1/k})^n$ for $n \in \mathbb{Z}$, which is again a k-valued function when k is relatively prime to n. Our primary concern will be the case $k = 2$. In this scenario, the non-principal branch can be expressed in terms of the principal branch as $-z^{1/2}$; the two branches of $z_m^{n/2}$ are then just given by $z^{n/2} := (z^{1/2})^n$ and $(-1)^n z^{n/2}$. The notation $\log(z)$ always refers to the principal branch of the logarithm, i.e. that branch for which $\log(1) = 0$. The notation $\Gamma(z)$ refers to the gamma function.