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Adaptive sparse-grid Gauss–Hermite filter

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HIGHLIGHTS

- A novel nonlinear filter based on sparse-grid quadrature is proposed.
- Accuracy of the GHF and SGHF is retained.
- Considerable reduction in computational burden of GHF and SGHF.
- A better trade-off between the accuracy and computational burden is achieved.

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ABSTRACT

In this paper, a new nonlinear filter based on sparse-grid quadrature method has been proposed. The proposed filter is named as adaptive sparse-grid Gauss–Hermite filter (ASGHF). Ordinary sparse-grid technique treats all the dimensions equally, whereas the ASGHF assigns a fewer number of points along the dimensions with lower nonlinearity. It uses adaptive tensor product to construct multidimensional points until a predefined error tolerance level is reached. The performance of the proposed filter is illustrated with two nonlinear filtering problems. Simulation results demonstrate that the new algorithm achieves a similar accuracy as compared to sparse-grid Gauss–Hermite filter (SGHF) and Gauss–Hermite filter (GHF) with a considerable reduction in computational load. Further, in the conventional GHF and SGHF, any increase in the accuracy level may result in an unacceptably high increase in the computational burden. However, in ASGHF, a little increase in estimation accuracy is possible with a limited increase in computational burden by varying the error tolerance level and the error weighting parameter. This enables the online estimator to operate near full efficiency with a predefined computational budget.

1. Introduction

In this article, we address the state estimation problem of a discrete nonlinear dynamic system with additive noise. The process and measurement model of the nonlinear system can, respectively, be defined as

$$\mathbf{x}_k = \phi(\mathbf{x}_{k-1}) + w_k \tag{1}$$

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and

$$y_k = \gamma(\mathbf{x}_k) + v_k, \tag{2}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ represents the unknown states of the system, $y_k \in \mathbb{R}^p$ denotes the measurement at any discrete time k. $\phi(\cdot)$ and $\gamma(\cdot)$ are known nonlinear functions. The process and measurement noises are represented by $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^p$, respectively. They are assumed to be uncorrelated and normally distributed with zero mean and covariance, Q and R, respectively.

Bayesian estimation framework is a widely employed method for addressing a filtering problem. In this framework, by using the measurement likelihood and the predicted motion of the unknown states, the posterior probability density functions (pdf) are computed [1].

During filtering of nonlinear systems, a set of intractable integrals appear and hence no optimal solution exists. In a widely accepted approach, the conditional pdfs are approximated as Gaussian and characterized with mean and covariance. Under this approach, a variety of filters like extended Kalman filter (EKF) [1], unscented Kalman filter (UKF) [2] and its extensions [3,4], cubature Kalman filter (CKF) [5] and its extension [6], and central difference filter (CDF) [7] are proposed. In a different approach, particle filter (PF) [8] is developed which approximates the true probability density function (pdf) with the help of particles and their assigned weights. Although the particle filter has high accuracy, its high computational burden restricts applicability in real time applications.

To achieve a higher accuracy under assigned computational budget, another Gaussian filter named Gauss–Hermite filter (GHF) [9] was introduced. GHF makes use of Gauss–Hermite quadrature rule for univariate systems. This univariate quadrature rule is extended to multidimensional domain by using the product rule, which in turn results in an exponential rise in multivariate quadrature points and hence suffers from the *curse of dimensionality* problem. This hinders the practical applicability of the filter for higher dimensional problems. We focus our study on decreasing the computational load of Gauss–Hermite filter without hampering its accuracy.

In an earlier approach, sparse-grid Gauss–Hermite filter (SGHF) was introduced which achieves similar accuracy as compared to the GHF, with reduced computational load [10]. In this technique, the univariate quadrature rule is extended to multivariate case with the help of Smolyak rule [11,12].

In this paper, we propose a novel approach which further reduces the computational burden of Gauss–Hermite filtering. The proposed method is named as adaptive sparse-grid Gauss–Hermite filter (ASGHF). The conventional sparse-grid method treats all the dimensions equally, by default, resulting in no immediate advantage for problems where the dimensions are of differing nonlinearity. But the proposed method uses adaptive sparse-grid technique [13] which automatically finds the dimensions with comparatively lower degree of nonlinearity and generate fewer points for approximation along them which further results in reduced computational cost.

Another advantage of using this method is that it provides a smooth relation between accuracy and computational burden. Unlike the GHF and the SGHF, a small rise in computational burden is possible in the proposed method for a corresponding small increase in the accuracy, by varying the predefined tolerance level and error weighting parameters. It enables the system to work with maximum efficiency possible within the allotted computational budget.

2. Sparse-grid Gauss–Hermite filter

While computing the mean and covariance matrix in an approximate Gaussian filter such as the GHF or SGHF, one encounters integrals of the following form:

$$\mathbf{I}_n(f^n(\mathbf{x})) = \int_{R^n} f^n(\mathbf{x})\mathcal{N}(\mathbf{x}; 0, \mathtt{I}_n)d\mathbf{x}, \tag{3}$$

where $f^n(\mathbf{x})$ is an n-dimensional nonlinear function and $\mathtt{I}_n$ is an n-dimensional unity matrix. In SGHF, this integral is approximated using Smolyak rule which makes use of difference formulas $\triangle_l f^1(\mathbf{x}) = (I_l - I_{l-1})f^1(\mathbf{x})$; $I_0 = 0$. Here I_l is a single dimensional quadrature rule with $(2l-1)$ univariate quadrature points. The set of points and weights for I_l can be generated using any of the moment matching method and Golub's Technique [9]. Using Smolyak rule [13],

$$\begin{aligned}\mathbf{I}_n(f^n(\mathbf{x})) &= \sum_{|\mathbb{I}|_{n,L} \le L+n-1} (\triangle_{l_1} \otimes \cdots \otimes \triangle_{l_n}) f^n(\mathbf{x}) \\ &= \sum_{\Xi \in N_q^n} (\triangle_{l_1} \otimes \cdots \otimes \triangle_{l_n}) f^n(\mathbf{x}),\end{aligned} \tag{4}$$

where $|\mathbb{I}|_{n,L}$ represents an n dimensional index set with accuracy level L and $\otimes$ stands for tensor product. $\Xi = [l_1 \quad l_2 \cdots l_n]^T$ represent a vector and N_q^n is defined as

$$\begin{aligned} N_q^n &= \left\{ \Xi : \sum_{j=1}^{n} l_j = n + q \right\} \qquad \text{for} \quad q \ge 0 \\ &= \varnothing \qquad\qquad\qquad\qquad\quad \text{for} \quad q < 0 \end{aligned}$$

where $\varnothing$ is null set and q is an integer *i.e.* $L - n \le q \le L - 1$.