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Optimal iterative methods for finding multiple roots of nonlinear equations using weight functions and dynamics ☆

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ABSTRACT

In this paper, we propose a family of iterative methods for finding multiple roots, with known multiplicity, by means of the introduction of four univariate weight functions. With the help of these weight functions, that play an important role in the development of higher order convergent iterative techniques, we are able to construct three-point eight-order optimal multiple-root finders. Also, numerical experiments have been applied to a number of test equations for different special schemes from this family satisfying the conditions given in the convergence analysis. We have also compared the basins of attraction of some proposed and known methods in order to check the wideness of the sets of converging initial points for each problem.

1. Introduction

Newton's method has been widely used to investigate simple or multiple zeros of a nonlinear equation. If the given function involves only a simple zero, then Newton's method converges quadratically to the exact solution provided that a proper initial guess is selected close to the exact solution. Newton's method has a drawback that it converges linearly when a given function has multiple roots. For a nonlinear equation of the form $f(x) = 0$, which involves multiple roots with multiplicity $m > 1$ a prior, modified Newton's method [1,2] is given as:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots \tag{1.1}$$

It efficiently locates the desired multiple zero with quadratic order convergence. The numerical scheme (1.1) is a second-order, one-point, optimal method on the basis of Kung–Traub's conjecture [3] which states that any multipoint method without memory can reach its convergence order of at most 2^{r-1} for r functional evaluations per iteration. In the recent past, many researchers like Li et al. [4] in (2009), Sharma and Sharma [5] and Li et al. [6] in (2010), Zhou et al. [7] in (2011), Sharifi et al. [8] in (2012), Soleymani et al. [9], Soleymani and Babajee [10], Liu and Zhou et al. [11] and Zhou et al. [12] in (2013), Thukral [13] in (2014), Behl et al. [14] and Hueso et al. [15] in (2015) and Behl et al. [16] in (2016) have presented optimal fourth-order iterative methods for multiple zeros. In addition, Li et al. [6] and Neta [17] presented optimal and

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non-optimal fourth-order iterative methods. Thukral [18] and Geum et al. [19,20] have been able to present sixth order convergent methods for finding multiple roots.

In 2013, Thukral [18] presented a multi-point iterative method with sixth-order convergence, which is given by

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - m\frac{f(x_n)}{f'(x_n)}\sum_{i=1}^{3} i\left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{i}{m}}, \\ x_{n+1} &= z_n - m\frac{f(x_n)}{f'(x_n)}\left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}\left\{\sum_{i=1}^{3} i\left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{i}{m}}\right\}^2. \end{aligned} \tag{1.2}$$

In 2015, Geum et al. [19], have given the following two-point, sixth-order iterative scheme, for $m > 1$:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - Q(p_n, s_n)\frac{f(y_n)}{f'(y_n)}, \end{aligned} \tag{1.3}$$

where $p_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$, $s_n = \sqrt[m-1]{\frac{f'(y_n)}{f'(x_n)}}$ and $Q : \mathbb{C}^2 \to \mathbb{C}$ is a holomorphic function in the neighborhood of origin $(0, 0)$.

In 2016, Geum et al. [20], have given a three-point iterative scheme with sixth-order convergence for multiple zeros involving weight function approach as follows:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \quad m > 1, \\ w_n &= x_n - mG(p_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - mK(p_n, t_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{1.4}$$

where $p_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$ and $s_n = \sqrt[m]{\frac{f(w_n)}{f(x_n)}}$. The weight function $G : \mathbb{C} \to \mathbb{C}$ is analytic in a neighborhood of 0 and $K : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic in a neighborhood of $(0, 0)$. All of above three schemes (1.2), (1.3), and (1.4) require four functional evaluations in order to produce sixth-order convergence with the efficiency index $6^{\frac{1}{4}} = 1.5650$.

Recently, in [21] Behl et al. have developed a family of optimal eighth order iterative methods given as:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \quad m > 1, \\ z_n &= y_n - u_n Q(h_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - u_n t_n G(h_n, t_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{1.5}$$

where $u_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$, $h_n = \frac{u_n}{a_1 + a_2 u_n}$, $t_n = \sqrt[m]{\frac{f(z_n)}{f(y_n)}}$ and $Q : \mathbb{C} \to \mathbb{C}$ is analytic in a neighborhood of 0 and $G : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic in the neighborhood of (0,0).

Motivated by the need to present a family of optimal higher order convergent methods for finding multiple roots, we present a family of optimal eighth order convergence method using only four function evaluations. Section 2 provides the methodology and convergence analysis, for the proposed optimal eight-order scheme. In Section 3, some special cases of the new scheme are considered. Section 4 includes the numerical experiments and comparisons of different multiple zero finders using test functions. Finally, conclusions are given in Section 5.

2. Construction of optimal scheme with eight-order convergence

Let us consider the following scheme involving univariate weight functions for solving the root-finding problem:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \quad m > 0, \\ z_n &= y_n - m u_n H(u_n)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - u_n P(u_n) G(v_n) L(w_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{2.1}$$