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Inertial extragradient algorithms for strongly pseudomonotone variational inequalities

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ABSTRACT

The purpose of this paper is to study and analyze two different kinds of inertial type iterative methods for solving variational inequality problems involving strongly pseudomonotone and Lipschitz continuous operators in Hilbert spaces. The projection method is used to design the algorithms which can be computed more easily. The construction of solution approximations and the proof of convergence of the algorithms are performed without prior knowledge of the modulus of strong pseudomonotonicity and the Lipschitz constant of cost operator. Instead of that, the algorithms use variable stepsize sequences which are diminishing and non-summable. The numerical behaviors of the proposed algorithms on a test problem are illustrated and compared with several previously known algorithms.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $A : H \to H$ be an operator. The variational inequality problem (VIP) [1–5] for A on C is to find $p \in C$ such that

$$\langle Ap, x - p \rangle \geq 0 \;\; \forall x \in C. \tag{VIP}$$

It is well known that the VIP is a central problem of nonlinear analysis. It is a useful mathematical model which unifies many important concepts in applied mathematics, such as necessary optimality conditions, network equilibrium problems, complementarity problems and systems of nonlinear equations, for instance [2,5–9]. Two notable and general directions for solving VIPs can be the regularized method and projection method. It is also emphasized that the first direction is often applied for the class of monotone operators. Regularized subproblem in this method is strongly monotone and its unique solution can be found more easily than solutions of the original problem. Regularized solutions can converge finitely or asymptotically to some solution of the original solution. For more general monotone VIPs, for example, pseudomonotone VIPs, which have been widely studied in recent years [10–12], the strong monotonicity of regularized subproblems can be destroyed. Thus, regularized methods cannot be applied in these cases.

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In this paper, we focus on the second direction: the projection methods which are well known for easier numerical computations. The early known projection method for VIPs is the gradient projection, after which, many other projection methods were developed including the extragradient method [13], the subgradient extragradient method [14–17], the projected reflected gradient methods [18–20] and others [21–27]. The aforementioned methods have been studied for VIPs which are monotone, strongly monotone or inverse strongly monotone. Moreover, a common point of these methods is that, in constructing solution approximations and establishing their convergence, the fixed or variable stepsizes often depend on factorials of cost operators, for instance, the strong monotone or Lipschitz constants of cost operators. This can make restrictions in applications because, in some cases, these constants can be unknown or difficult to approximate.

Recently, based on known projection methods, the authors in [28–33] introduced new methods for strongly pseudomonotone and Lipschitzian VIPs where the stepsizes are variable and independent from the strongly pseudomonotone and Lipschitz constants of cost operators. It is worth mentioning that the class of strongly pseudomonotone VIPs properly contains the class of strongly monotone VIPs.

Now, let us mention an inertial-type algorithm. Based on the heavy ball methods of the two-order time dynamical system, Polyak [34] firstly proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. The inertial algorithm is a two-step iterative method, and the next iterate is defined by making use of the previous two iterates and it can be regarded as a procedure of speeding up the convergence properties, see [34–38]. Recently, a lot of researchers constructed fast iterative algorithms by using inertial extrapolation, including inertial forward–backward splitting methods [39–42], inertial Douglas–Rachford splitting method [43], inertial ADMM [44,45], inertial forward–backward–forward method [46], inertial proximal-extragradient method [47], inertial contraction method [48], inertial subgradient extragradient method [49], and inertial Mann method [50].

Motivated by the presented results, in this paper, we propose two algorithms for solving strongly pseudomonotone and Lipschitzian VIPs. First, we propose an inertial subgradient extragradient method (ISEGM). This method works and bases on the subgradient extragradient method [14] and the inertial method [35]. Second, we combine the inertial method and the Tseng's extragradient [51] to introduce an inertial Tseng's extragradient method (ITEGM). In these methods, a projection onto a feasible set and two values of cost operator need to be computed per each iteration. Finally, we consider a test problem and illustrate the numerical behaviors of the algorithms in [30,52,53] and compare them with the two algorithms presented in this paper.

This paper is organized as follows: In Section 2, we recall some definitions and preliminary results for further use. Section 3 deals with proposing the algorithms and analyzing their convergence. Finally, in Section 4 we perform several numerical experiments to illustrate the computational performance of the proposed algorithm over several previously known algorithms.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \le \|x - y\| \quad \forall y \in C$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1 (*[20,54]*)**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x \iff \langle x - z, z - y\rangle \ge 0 \quad \forall y \in C$.*

Lemma 2.2 (*[20,54]*)**.** *Let C be a closed and convex subset in a real Hilbert space H, $x \in H$. Then*
(i) $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle \quad \forall y \in C$;
(ii) $\|P_C x - y\|^2 \le \|x - y\|^2 - \|x - P_C x\|^2 \quad \forall y \in C$.

For properties of the metric projection, the interested reader could be referred to Section 3 in [54]. We present some concepts of monotonicity of an operator.

Definition 2.1. An operator $A : H \to H$ is said to be:
(i) *strongly monotone* on C if there exists $\gamma > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \gamma\|x - y\|^2 \quad \forall x, y \in C;$$

(ii) *monotone* on C if

$$\langle Ax - Ay, x - y\rangle \ge 0 \quad \forall x, y \in C;$$

(iii) *strongly pseudomonotone* on C if there exists $\gamma > 0$ such that

$$\langle Ax, y - x\rangle \ge 0 \Longrightarrow \langle Ay, x - y\rangle \le -\gamma\|x - y\|^2 \quad \forall x, y \in C.$$

(iv) *pseudomonotone* on C if

$$\langle Ax, y - x\rangle \ge 0 \Longrightarrow \langle Ay, x - y\rangle \le 0 \quad \forall x, y \in C.$$