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New discretization schemes for time-harmonic Maxwell equations by weak Galerkin finite element methods

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ABSTRACT

This paper introduces new discretization schemes for time-harmonic Maxwell equations in a connected domain by using the weak Galerkin (WG) finite element method. The corresponding WG algorithms are analyzed for their stability and convergence. Error estimates of optimal order in various discrete Sobolev norms are established for the resulting finite element approximations.

1. Introduction

This paper is concerned with new developments of numerical methods for time-harmonic Maxwell equations. The time-harmonic Maxwell equations are coupled magnetic and electric equations given by

$$\begin{aligned}
\nabla \times \mathbf{E} =& -\frac{\partial \mathbf{B}}{\partial t}, && \text{in } \Omega,\\
\nabla \times \mathbf{H} =& \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, && \text{in } \Omega,\\
\nabla \cdot \mathbf{D} =& \rho, && \text{in } \Omega,\\
\nabla \cdot \mathbf{B} =& 0, && \text{in } \Omega,
\end{aligned} \tag{1.1}$$

with the constitutive relations:

$$\mathbf{B} = \mu \mathbf{H}, \mathbf{j} = \sigma \mathbf{E} + \mathbf{j}_e, \mathbf{D} = \varepsilon \mathbf{E},$$

where Ω is an open bounded and connected domain in $\mathbb{R}^d (d = 2, 3)$ with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. Here, **E** is the electric field intensity, **B** is the magnetic flux density, **H** is the magnetic field intensity, **D** is the electric displacement

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flux density, $\mathbf{j}$ is the electric current density, $\mu = \{\mu_{ij}(\mathbf{x})\}_{d\times d}$ is called permeability, ρ is the charge density, $\mathbf{j}_e$ is the external current density, σ is real-valued and is known as the electric conductivity, and $\varepsilon = \{\varepsilon_{ij}(\mathbf{x})\}_{d\times d}$ is the material parameter which is called permittivity. Additionally, μ, ε are real-valued, symmetric, uniformly positive definite matrices in the domain Ω. We assume that μ, ε and σ are piecewise smooth functions in the domain Ω.

For time-harmonic fields, where the time dependence is assumed to be harmonic, i.e., $\exp(i\omega t)$, using the constitutive relations, the Maxwell equations (1.1) can be rewritten for the Fourier transform of the fields as (see [1] for details)

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}, \qquad \text{in } \Omega, \tag{1.2}$$

$$\nabla \times \mathbf{H} = i\omega\varepsilon\mathbf{E} + \sigma\mathbf{E} + \mathbf{j}_e, \qquad \text{in } \Omega, \tag{1.3}$$

$$\nabla \cdot (\varepsilon\mathbf{E}) = \rho, \qquad \text{in } \Omega, \tag{1.4}$$

$$\nabla \cdot (\mu\mathbf{H}) = 0, \qquad \text{in } \Omega, \tag{1.5}$$

where ω is a constant in the domain Ω.

In the past several decades, the Maxwell equations have been extensively investigated by many researchers. H(curl) conforming finite element method was first introduced by J. Nédélec [2] and was further developed by P. Monk [3]. Houston, Perugia and Schotzau [4–8] have developed discontinuous Galerkin (DG) finite element methods for the Maxwell equations. Particularly in [6], a mixed DG formulation for the Maxwell equations was introduced and analyzed. Recently, a weakly over-penalized symmetric interior penalty method [9] has been introduced and analyzed by S. Brenner, F. Li and L. Sung. There are also many other numerical methods developed to discretize the Maxwell equations.

Recently, WG method is emerging as an efficient finite element technique for partial differential equations. The WG finite element method was first introduced in [10,11] for second order elliptic equations and the idea was subsequently further developed for several other model PDEs [12–17]. The key idea of WG method is to use weak functions and their corresponding discrete weak derivatives in existing variational forms. WG method is highly flexible and robust by allowing the use of discontinuous piecewise polynomials and finite element partitions with arbitrary shape of polygons/polyhedra, and the method is parameter free and absolutely stable. WG finite element method has been applied to time-harmonic Maxwell equations in [18], yielding a numerical method that has optimal order of convergence in certain discrete norms.

The goal of this paper is to present a new WG finite element method for the time-harmonic Maxwell equations (1.2)–(1.5) in a connected domain with heterogeneous media, which covers more cases compared with the model problem considered in [18]. In particular, we formulate the time-harmonic Maxwell equations (1.2)–(1.5) into two variational problems with complex coefficients; see (2.3) and (2.4) for details. Each of the variational problems is then discretized by using the weak Galerkin finite element method. The main difficulty in the design of numerical methods for (2.3) and (2.4) lies in the fact that the terms $\nabla \cdot (\varepsilon\mathbf{E})$ and $\nabla \cdot (\mu\mathbf{H})$ require the continuity of $\varepsilon\mathbf{E}$ and $\mu\mathbf{H}$ in the normal direction of all interior interfaces, respectively. Consequently, the usual $H(div)$ or $H(curl)$ conforming elements are not applicable in this practice. This paper shows that the weak Galerkin finite element method offers an ideal solution, as the continuity can be relaxed by a weak continuity implemented through a carefully chosen stabilizer.

The paper is organized as follows. In Section 2, we shall derive two variational problems: one for the electric field intensity and the other for the magnetic field intensity. These variational problems form the basis of the weak Galerkin finite element methods of this paper. In Section 3, we shall briefly review the discrete weak divergence and the discrete weak curl operators which are necessary in weak Galerkin. In Section 4, we describe how the weak Galerkin finite element algorithms are formulated. Section 5 is devoted to a verification of some stability conditions for the resulting WG algorithms. In particular, it is shown in this section that the WG algorithms have one and only one solution. In Section 6, we derive some error equations for our WG algorithms. Finally in Section 7, we establish some optimal order error estimates for the WG finite element approximations.

Throughout the paper, we will follow the usual notations for Sobolev spaces and norms [19]. For any open bounded domain $D \subset \mathbb{R}^d (d = 2, 3)$ with Lipschitz continuous boundary, we use $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ to denote the norm and seminorm in the Sobolev space $H^s(D)$ for any $s \geq 0$, respectively. The inner product in $H^s(D)$ is denoted by $(\cdot,\cdot)_{s,D}$. The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively.

We introduce the following Sobolev space

$$H(\mathrm{div}_\varepsilon; D) = \{\mathbf{v} \in [L^2(D)]^d : \nabla \cdot (\varepsilon\mathbf{v}) \in L^2(D)\},$$

with norm given by

$$\|\mathbf{v}\|_{H(\mathrm{div}_\varepsilon; D)} = (\|\mathbf{v}\|_D^2 + \|\nabla \cdot (\varepsilon\mathbf{v})\|_D^2)^{\frac{1}{2}},$$

where $\nabla \cdot (\varepsilon\mathbf{v})$ is the divergence of $\varepsilon\mathbf{v}$. Any $\mathbf{v} \in H(\mathrm{div}_\varepsilon; D)$ can be assigned a trace for the normal component of $\varepsilon\mathbf{v}$ on the boundary. Denote the subspace of $H(\mathrm{div}_\varepsilon; D)$ with vanishing trace in the normal component by

$$H_0(\mathrm{div}_\varepsilon; D) = \{\mathbf{v} \in H(\mathrm{div}_\varepsilon; D) : (\varepsilon\mathbf{v}) \cdot \mathbf{n}|_{\partial D} = 0\}.$$

When $\varepsilon = I$ is the identity matrix, the spaces $H(\mathrm{div}_\varepsilon; D)$ and $H_0(\mathrm{div}_\varepsilon; D)$ are denoted as $H(\mathrm{div}; D)$ and $H_0(\mathrm{div}; D)$, respectively.