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# Weighted finite element method for the Stokes problem with corner singularity

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## HIGHLIGHTS

- An $R_\nu$-generalized solution to the Stokes problem with corner singularity is introduced.
- The weighted analogue of the LBB condition is proved.
- A new weighted finite element method is constructed.
- An approximate solution converges to an exact one with $O(h)$ rate in $W^1_{2,\nu}(\Omega)$ seminorm.

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## ABSTRACT

In this paper we introduce the notion of $R_\nu$-generalized solution to the Stokes problem with singularity in a non-convex polygonal domain with one reentrant corner of $\frac{3\pi}{2}$ on its boundary. The weighted analogue of the Ladyzhenskaya–Babuška–Brezzi condition is proved. A new weighted finite element method is constructed. Results of numerical experiments have shown the efficiency of the method.



## 1. Introduction

The weak solution of Maxwell equations considered in a 2$D$ polygonal domain with reentrant corner on the boundary does not belong to the Sobolev space $W^1_2(\Omega)$. Such a problem is called a boundary value problem with strong singularity. For the Lamé system, for an example, in a domain with a reentrant corner it is possible to define a weak solution in the space $W^1_2(\Omega)$, but it does not belong to the space $W^2_2(\Omega)$. Such problem is called a problem with weak singularity.

According to the principle of coordinated estimates, the approximate solution to these problems by the classical finite difference methods and finite element methods converge to the exact one with a rate substantially smaller than one. In [1,2] it was proposed to define the solution of elliptic boundary value problems and Maxwell equations with strong singularity as an $R_\nu$-generalized one. Such a new conception of solution allows to construct weighted finite element methods with first-order convergence rate estimate of the approximate solution to the $R_\nu$-generalized one in the norms of the weighted Sobolev spaces.

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In this paper we present our method for the Stokes problem. It is not a secret that the efficient numerical solution of problems in fluid mechanics is of significant engineering interest. There are basically three reasons why the finite element discretization of such problem turns out to be difficult.

Firstly, in the presence of reentrant corners in the domain with interior angle $\varphi$, $\varphi \in (\pi, 2\pi)$, the solution of the problem is singular even though the input data are smooth. The two-dimensional flow of a viscous fluid near the corner was first examined in [3]. It is well known that the generalized solution of the Stokes problem: the velocity components and pressure in a two-dimensional domain $\Omega$ with a boundary containing a reentrant angle, does not belong to $W_2^2(\Omega)$ and $W_2^1(\Omega)$ respectively (see e.g., [4]). Therefore, the approximate solution produced by standard finite element or finite difference schemes converges to a generalized solution no faster that at an $\mathcal{O}(h^\alpha)$ rate in the seminorm of the space $W_2^1(\Omega)$, where $\alpha < 1$ for the velocity components (see [5]). In this case the so-called pollution effect can be observed in standard Sobolev and even in weighted Sobolev norms [6]. More recent results on the regularity theory and finite element approximations on domains with reentrant corners can be found in [7–9] and the references therein. By using special methods for extracting the singular part of the solution near corner points or applying grids refined towards the singularity point, it is possible to construct first-order accurate finite element schemes (see e.g., [10]). Secondly, the design of inf–sup stable methods for the velocity and pressure spaces pairs [11].

Thirdly, the spaces enforce mass conservation strongly. Satisfying this criterion leads to more physically relevant solutions, decouples the pressure error from the velocity error, and removes possible instabilities that can arise from poor discrete mass conservation [12]. The specific element pair to achieve pointwise mass conservation of the discrete solution is the Scott–Vogelius element pair [13,14].

In the present paper we introduce the notion of $R_\nu$-generalized solution [15–17] of the Stokes problem with a singularity due to a reentrant corner of $\frac{3\pi}{2}$ on its boundary. It is well known that for the well-posedness of the incompressible flow problem, the Ladyzhenskaya–Babuška–Brezzi (LBB) condition plays important role. We formulate and prove the weighted analogue of this condition. Then, we construct the weighted finite element method (see [1,2,18–20]) based on the conception of $R_\nu$-generalized solution [15–17] and mass conservation Scott–Vogelius element pair (mesh created as a barycenter refinement). Numerical experiments of the model problem have shown that the approximate $R_\nu$-generalized solution converges to the exact one (velocity) with the rate $\mathcal{O}(h)$ in $\mathbf{W}^1_{2,\nu}(\Omega)$ seminorm. Another advantage of this method is the simplicity of the solution determination which is an additional benefit for the numerical experiments.

The structure of the paper is as follows. In Section 2 we introduce the necessary notations and prove auxiliary statements. In Section 3 we define the $R_\nu$-generalized solution of the Stokes problem with corner singularity, formulate and prove the weighted analogue of the LBB condition. In Section 4 we describe the proposed weighted finite element method using Scott–Vogelius element pair ($k = 2$), which is very interesting from the mass conservation point of view. In Section 5 we construct an iterative process with a block preconditioning matrix. In Section 6 we present and discuss the results of numerical experiments. Finally, some concluding remarks are given in Section 7.

## 2. Notation and auxiliary statements

Let $\mathbf{R}^2$ denote the two-dimensional Euclidean space, $\mathbf{x} = (x_1, x_2)$ be its arbitrary element, $\|\mathbf{x}\| = \left(x_1^2 + x_2^2\right)^{1/2}$ and $d\mathbf{x} = dx_1\,dx_2$. Let $\Omega \subset \mathbf{R}^2$ be a bounded non-convex polygonal domain with a boundary $\Gamma$ containing a reentrant angle with its vertex placed at the origin, and let $\bar\Omega$ be a closure of $\Omega$, i.e. $\bar\Omega = \Omega \cup \Gamma$. Denote by $\Omega'_\delta = \{\mathbf{x} \in \bar\Omega : \|\mathbf{x}\| \le \delta < 1, \delta > 0\}$ the part of a $\delta$-neighborhood of the point (0, 0) that lies in $\bar\Omega$. Define a weight function $\rho(\mathbf{x})$, such that $\rho(\mathbf{x}) = \begin{cases} \|\mathbf{x}\|, & \mathbf{x} \in \Omega'_\delta, \\ \delta, & \mathbf{x} \in \bar\Omega \setminus \Omega'_\delta. \end{cases}$

Let $L_{2,\beta}(\Omega)$ denote the weighted space of functions with bounded norm

$$\|v\|_{L_{2,\beta}(\Omega)} = \left(\int_\Omega \rho^{2\beta}\, v^2 d\mathbf{x}\right)^{1/2}$$

and $W^1_{2,\beta}(\Omega)$ denote the weighted space of functions with bounded norm

$$\|v\|_{W^1_{2,\beta}(\Omega)} = \left(\sum_{|m|\le 1} \|\rho^\beta |D^m v|\|^2_{L_2(\Omega)}\right)^{1/2}, \tag{1}$$

where $D^m v = \frac{\partial^{|m|} v}{\partial x_1^{m_1}\, \partial x_2^{m_2}}$, $|m| = m_1 + m_2, m_i \ge 0, i = 1, 2$ — integer, $\beta$ is nonnegative real number. Let $|v|_{W^1_{2,\beta}(\Omega)}$ be a seminorm of a function $v$ in the space $W^1_{2,\beta}(\Omega)$: $|v|_{W^1_{2,\beta}(\Omega)} \equiv \|\nabla v\|_{L_{2,\beta}(\Omega)} = \left(\sum_{|m|=1} \|\rho^\beta(\mathbf{x}) |D^m v(\mathbf{x})|\|^2_{L_2(\Omega)}\right)^{1/2}$. For vector functions $\mathbf{v} = (v_1, v_2)$ we define weighted spaces $\mathbf{L}_{2,\beta}(\Omega)$ and $\mathbf{W}^1_{2,\beta}(\Omega)$ with norms $\|\mathbf{v}\|_{\mathbf{L}_{2,\beta}(\Omega)} = \left(\sum_{i=1}^2 \|v_i\|^2_{L_{2,\beta}(\Omega)}\right)^{1/2}$ and $\|\mathbf{v}\|_{\mathbf{W}^1_{2,\beta}(\Omega)} = \left(\sum_{i=1}^2 \|v_i\|^2_{W^1_{2,\beta}(\Omega)}\right)^{1/2}$ respectively. Let $|\mathbf{v}|_{\mathbf{W}^1_{2,\beta}(\Omega)}$ be a seminorm of a vector function $\mathbf{v}$ in the space $\mathbf{W}^1_{2,\beta}(\Omega)$: $|\mathbf{v}|_{\mathbf{W}^1_{2,\beta}(\Omega)} \equiv \|\nabla \mathbf{v}\|_{\mathbf{L}_{2,\beta}(\Omega)} = \left(\sum_{i=1}^2 \left(\sum_{|m|=1} \|\rho^\beta |D^m v_i|\|^2_{L_2(\Omega)}\right)\right)^{1/2}$.