ARTICLE IN PRESS

Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Flux reconstructions in the Lehmann–Goerisch method for lower bounds on eigenvalues

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ARTICLE INFO

Article history:
Received 30 May 2017
Received in revised form 20 February 2018

MSC:
65N25
65N30
65N15

Keywords:
Eigenproblem
Guaranteed
Symmetric
Elliptic operators
Finite element method
Conforming

ABSTRACT

The standard application of the Lehmann–Goerisch method for lower bounds on eigenvalues of symmetric elliptic second-order partial differential operators relies on determination of fluxes $\tilde{\boldsymbol{\sigma}}_i$ that approximate co-gradients of exact eigenfunctions scaled by corresponding eigenvalues. Fluxes $\tilde{\boldsymbol{\sigma}}_i$ are usually computed by solving a global saddle point problem with mixed finite element methods. In this paper we propose a simpler global problem that yields fluxes $\tilde{\boldsymbol{\sigma}}_i$ of the same quality. The simplified problem is smaller, it is positive definite, and any $\boldsymbol{H}(\mathrm{div}, \Omega)$ conforming finite elements, such as Raviart–Thomas elements, can be used for its solution. In addition, these global problems can be split into a number of independent local problems on patches, which allows for trivial parallelization. The computational performance of these approaches is illustrated by numerical examples for Laplace and Steklov type eigenvalue problems. These examples also show that local flux reconstructions enable computation of lower bounds on eigenvalues on considerably finer meshes than the traditional global reconstructions.

1. Introduction

Methods for lower bounds on eigenvalues of symmetric elliptic partial differential operators attract growing attention in the last years [1–13]. The Lehmann–Goerisch method stems from a long history of development [14–16] and it is one of the most advanced methods. It is based on the Lehmann method [17,18] and the $(\boldsymbol{X}, \mathcal{B}, T)$ concept of Goerisch [19]. Practically, this method relies on conforming approximations of eigenfunctions of interest, subsequent flux reconstructions, and an *a priori* known (rough) lower bound of certain eigenvalue. In this paper we concentrate on flux reconstructions that approximate co-gradients of approximate eigenfunctions scaled by corresponding eigenvalues.

From the computational point of view, the flux reconstruction is usually obtained by solving a global saddle point problem [20]. This problem is considerably larger than the original eigenvalue problem, its saddle point structure brings technical difficulties, and for large problems it is a bottleneck of this approach.

Therefore, we propose to reconstruct the fluxes by solving a smaller (in terms of degrees of freedom) and simpler problem. The simpler problem provides the flux reconstruction of the same quality and in addition it is positive definite (meaning that the corresponding matrix is positive definite). Thus, it can be solved by any $\boldsymbol{H}(\mathrm{div}, \Omega)$ conforming finite elements as opposed to the original saddle point problem, where a suitable mixed finite element method has to be employed. Despite these advantages, even the simpler problem for fluxes is considerably larger than the eigenvalue problem itself. Therefore, we utilize the idea of [21–23] and propose localized versions of both the saddle point and simpler problems. Localized versions are based on solving independent small local problems on patches of elements and their accuracy is competitive with global problems. The main advantage of the localized problems lies in the fact that they are independent and can be solved in

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https://doi.org/10.1016/j.cam.2018.02.034
0377-0427/

parallel. Their memory requirements are low and they enable us to compute lower bounds on eigenvalues for considerably finer meshes than the traditional global flux reconstructions.

The main goal of this paper is to provide the flux reconstruction procedures for a general eigenvalue problem: find $\lambda_i > 0$ and $u_i \neq 0$ such that

$$\begin{aligned} -\operatorname{div}(\mathcal{A}\nabla u_i) + cu_i &= \lambda_i \beta_1 u_i \quad \text{in } \Omega, \\ (\mathcal{A}\nabla u_i)\cdot \boldsymbol{n}_\Omega + \alpha u_i &= \lambda_i \beta_2 u_i \quad \text{on } \Gamma_{\mathrm{N}}, \\ u_i &= 0 \qquad \text{on } \Gamma_{\mathrm{D}}, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is an open Lipschitz domain, d a dimension, Γ_{D} and Γ_{N} are two relatively open components of $\partial\Omega$ such that $\overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}} = \partial\Omega$ and $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$, and $\boldsymbol{n}_\Omega$ is the unit outward facing normal vector to the boundary $\partial\Omega$. Note that specific choices of parameters in problem (1) yield to the standard eigenvalue problems such as the Laplace eigenvalue problem and Steklov eigenvalue problem.

However, in order to explain the main idea without technicalities, we first consider the Laplace eigenvalue problem, see Sections 2–3. The following sections deal with the general eigenvalue problem. Section 4, in particular, shifts the eigenvalue problem (1) and briefly presents its well-posedness and finite element discretization. Section 5 introduces the Lehmann–Goerisch method and the global mixed finite element problem for the flux reconstruction. Section 6 analyses the Lehmann–Goerisch method and derives the simplified global problem for the flux reconstruction. Section 7 presents local versions of these global problems and transforms them to a series of independent problems on patches of elements. Sections 8–9 compare the accuracy and computational performance of the global and local flux reconstructions for the Laplace and Steklov-type eigenvalue problem on a dumbbell shaped domain. Finally, Section 10 draws conclusions.

2. The Lehmann–Goerisch method for Laplace eigenvalue problem

We first describe how to obtain lower bounds on eigenvalues by the Lehmann–Goerisch method for the special case of the Laplace eigenvalue problem. We seek eigenvalues $\lambda_i > 0$ and eigenfunctions $u_i \neq 0$ such that

$$\begin{aligned} -\Delta u_i &= \lambda_i u_i \quad \text{in } \Omega, \\ u_i &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

The weak formulation of this problem is posed in the Sobolev space $V = H^1_0(\Omega)$ consisting of $H^1(\Omega)$ functions with vanishing traces on $\partial\Omega$ and reads as follows: find eigenvalues $\lambda_i > 0$ and eigenfunctions $u_i \in V \setminus \{0\}$ such that

$$(\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V, \tag{3}$$

where $(\cdot,\cdot)$ stands for the $L^2(\Omega)$ inner product. This problem is well posed and possesses a countable sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, see e.g. [24,25].

In order to discretize problem (3) by the standard conforming finite element method, we consider Ω to be a polytope. We introduce a standard simplicial mesh $\mathcal{T}_h$ in Ω and define the lowest-order finite element space

$$V_h = \{v_h \in V : v_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\}, \tag{4}$$

where $P_1(K)$ is the space of affine functions on the simplex K. The finite element approximation of problem (3) corresponds to the finite dimensional problem of seeking eigenvalues $\Lambda_{h,i} \in \mathbb{R}$ and eigenfunctions $u_{h,i} \in V_h \setminus \{0\}$ such that

$$(\nabla u_{h,i}, \nabla v_h) = \Lambda_{h,i}(u_{h,i}, v_h) \quad \forall v_h \in V_h. \tag{5}$$

Discrete eigenvalues are naturally sorted in ascending order: $0 < \Lambda_{h,1} \leq \Lambda_{h,2} \leq \cdots \leq \Lambda_{h,N}$, where $N = \dim V_h$.

It is well known that $\Lambda_{h,i}$ approximates λ_i from above and that the order of convergence of the finite element approximation $\Lambda_{h,i}$ is optimal (quadratic in the case of no singularities) [24,25]. The Lehmann–Goerisch method enables us to compute approximations of λ_i from below with the same order of convergence. The idea of this method is summarized in [20, Theorem 2.1]. For the readers' convenience we recall this theorem here. Note that $\boldsymbol{W} = \boldsymbol{H}(\operatorname{div}, \Omega)$ denotes the standard space of square integrable vector fields with square integrable divergence.

Theorem 2.1 (*Behnke, Mertins, Plum, Wieners*). *Let $\tilde{u}_i \in V$, $\tilde{\boldsymbol{\sigma}}_i \in \boldsymbol{W}$, $i = 1, 2, \ldots, n$, and $\rho > 0$, $\gamma > 0$ be arbitrary. Define matrices $\boldsymbol{M}, \boldsymbol{N} \in \mathbb{R}^{n\times n}$ with entries*

$$\begin{aligned} \boldsymbol{M}_{ij} &= (\nabla\tilde{u}_i, \nabla\tilde{u}_j) + (\gamma - \rho)(\tilde{u}_i, \tilde{u}_j), \\ \boldsymbol{N}_{ij} &= (\nabla\tilde{u}_i, \nabla\tilde{u}_j) + (\gamma - 2\rho)(\tilde{u}_i, \tilde{u}_j) + \rho^2(\tilde{\boldsymbol{\sigma}}_i, \tilde{\boldsymbol{\sigma}}_j) + (\rho^2/\gamma)(\tilde{u}_i + \operatorname{div}\tilde{\boldsymbol{\sigma}}_i, \tilde{u}_j + \operatorname{div}\tilde{\boldsymbol{\sigma}}_j). \end{aligned}$$

Suppose that the matrix $\boldsymbol{N}$ is positive definite and that

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$$

Please cite this article in press as: T. Vejchodský, Flux reconstructions in the Lehmann–Goerisch method for lower bounds on eigenvalues, Journal of Computational and Applied Mathematics (2018), https://doi.org/10.1016/j.cam.2018.02.034.