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Convergence and stability of the compensated split-step theta method for stochastic differential equations with piecewise continuous arguments driven by Poisson random measure[☆]

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ABSTRACT

This paper deals with the numerical solutions of stochastic differential equations with piecewise continuous arguments (SDEPCAs) driven by Poisson random measure in which the coefficients are highly nonlinear. It is shown that the compensated split-step theta (CSST) method with $\theta \in [0, 1]$ is strongly convergent in pth$(p \geq 2)$ moment under some polynomially Lipschitz continuous conditions. It is also obtained that the convergence order is close to $\frac{1}{p}$. In terms of the stability, it is proved that the CSST method with $\theta \in (\frac{1}{2}, 1]$ reproduces the exponential mean square stability of the underlying system under the monotone condition and some restrictions on the step-size. Without any restriction on the step-size, there exists $\theta^* \in (\frac{1}{2}, 1]$ such that the CSST method with $\theta \in (\theta^*, 1]$ is exponentially stable in mean square. Moreover, if the drift and jump coefficients satisfy the linear growth condition, the CSST method with $\theta \in [0, \frac{1}{2}]$ also preserves the exponential mean square stability. Some numerical simulations are presented to verify the conclusions.

1. Introduction

Stochastic differential equations with piecewise continuous arguments (SDEPCAs) play an important role in many fields including control theory and neural networks, such as [1–4]. However, all models mentioned above are driven by Brownian motions, and Brownian motions are not always appropriate to interpret some real phenomena, such as abrupt pulses or extreme events. Taking these abrupt changes into consideration, a more natural mathematical model other than purely Brownian perturbations are required. In particular, the Poisson perturbations with jumps are of significance for SDEPCAs to describe abrupt changes. In this paper, we deal with the following d-dimensional SDEPCAs driven by Poisson random measure

$$\begin{cases} dx(t) = \mu(x(t^-), x([t^-]))dt + \sigma(x(t^-), x([t^-]))dB(t) + \int_Z \gamma(x(t^-), x([t^-]), \nu)N(dt, d\nu), \\ x(0) = \xi, \end{cases} \tag{1.1}$$

where $t \in [0, T]$, $x(t^-) = \lim_{s\to t^-} x(s)$, $\mu : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$, $\gamma : \mathbb{R}^d \times \mathbb{R}^d \times Z \to \mathbb{R}^d$ and for every $p > 0$ there exists $L(p) > 0$ such that $\mathbb{E}\|\xi\|^p < L(p)$. $B(t)$ is an r-dimensional Brownian motion and $[t]$ denotes the greatest-integer

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part of t. SDEs driven by Poisson random measure and SDDEs driven by Poisson random measure have been well investigated in [5–8].

SDEPCAs driven by Poisson random measure fit within the general paradigm of delay differential or functional differential equations driven by Poisson random measure, but the delays are discontinuous and the delay term $[t]$ is different from $t-\tau$. On one hand, SDEPCAs driven by Poisson random measure are nonautonomous, since the delays vary with t. On the other hand, the solutions are determined by a finite set initial data, rather than by an initial function, as in the case of general functional differential equations driven by Poisson random measure. In addition, continuity of a solution at a point joining any two consecutive intervals implies recursion relations for the solution at such points. Therefore, Eq. (1.1) represents a hybrid of continuous and discrete dynamical systems and combines the properties of both differential and difference equations. Based on these characteristics, it is meaningful to consider SDEPCAs driven by Poisson random measure.

Since there is in general no explicit solutions to an SDEPCA driven by Poisson random measure, numerical solutions are required in practice. The convergence and stability of numerical methods are well investigated for stochastic differential equations (SDEs) driven by Poisson random measure, such as [9–14]. But the results above require that the coefficients satisfy the global Lipschitz as well as the linear growth conditions. In practice, many systems fail to satisfy the linear growth condition. According to Hutzenthaler and his partners' results in [15], the Euler method may be not convergent for a class of SDEs driven by Poisson random measure with superlinearly growing coefficients.

To overcome this difficulty, Dareiotis et al. [16] obtained that the explicit tamed Euler method is strongly convergent in pth ($p \geq 2$) moment for SDEs with random coefficients driven by Lévy noise by assuming one-sided local Lipschitz conditions on drift coefficients and local Lipschitz conditions on both diffusion and jump coefficients. What is more, the authors applied this taming technique to solve SDDEs driven by Lévy noise. Before this paper, Bao et al. [17] investigated the strong convergence rate of the Euler method for stochastic functional differential equations (SFDEs) with jumps under a local Lipschitz condition. Later on, Bao and Yuan [18] obtained the strong convergence rate of the Euler method for SDDEs with jumps in which the coefficients are highly nonlinear with respect to delay variables.

However, the explicit methods are not so well in terms of stability. Thus based on its advantages on stability, implicit methods attract a lot of attention to overcome the difficulty above. There are a lot of references on implicit methods for SDEs (see [19–21]). There are also many works on implicit methods for SDEs with jumps and SDDEs with jumps where the coefficients are non-globally Lipschitz continuous, such as [22–24]. Since Huang [25] pointed out the implicit split-step θ (SST) method, many scholars are devoted to investigate this method due to its advantage on stability (see [26–28]). Our aims are to investigate the strong convergence and exponential stability of the compensated split-step θ (CSST) method for SDEPCAs driven by Poisson random measure under some local Lipschitz conditions. To our best knowledge, it is the first work on SDEPCAs. The conditions which the SDEPCAs satisfy are much weaker than global Lipschitz conditions. The results are as follows.

- The CSST method with $\theta \in [0, 1]$ is pth ($p \geq 2$) moment bounded when the drift, diffusion and jump coefficients all satisfy the linear growth conditions in non-delay terms whereas superlinear growth conditions in delay arguments.
- Based on the pth moment boundedness, we show that the CSST method with $\theta \in [0, 1]$ is strongly convergent in pth ($p \geq 2$) moment, when the drift coefficients are one-sided Lipschitz continuous in non-delay terms and polynomially Lipschitz continuous in delay arguments while only the delay terms are polynomially Lipschitz continuous in both diffusion and jump coefficients. The convergence order is close to $\frac{1}{p}$, which is consistent with the existing work.
- From the aspect of stability, the CSST method with $\theta \in (\frac{1}{2}, 1]$ reproduces the exponential mean square stability of the underlying systems under the monotone condition and some restrictions on the step-size.
- Without any restriction on the step-size, there exists $\theta^* \in (\frac{1}{2}, 1]$ such that the CSST method with $\theta \in (\theta^*, 1]$ is exponentially stable in mean square.
- Moreover, if the drift and jump coefficients satisfy the linear growth conditions, the CSST method with $\theta \in [0, \frac{1}{2}]$ also preserves the exponential mean square stability.

An outline of this paper is as follows. In Section 2, some preliminary notations and the CSST method for SDEPCAs driven by Poisson random measure are introduced. In Section 3, if the drift coefficients satisfy the one-sided Lipschitz conditions with respect to non-delay terms and are polynomially Lipschitz continuous in delay arguments, while both the diffusion and jump coefficients satisfy the polynomial Lipschitz conditions with respect to delay terms only but satisfy the linear growth conditions in non-delay terms, the pth ($p \geq 2$) moment convergence of the CSST method is obtained. In addition, the pth moment convergence order is shown to be close to $\frac{1}{p}$. In Section 4, both the underlying system and the CSST method are proved to be exponentially mean square stable when the coefficients satisfy the monotone condition. The numerical examples are provided to explain the analytical results in Section 5.

2. Preliminary notations and the CSST method

Throughout this paper, we use the following notations unless otherwise specified. Let $\|x\|$ denotes the Euclidean vector norm in $\mathbb{R}^n$ and $\langle x, y\rangle$ denotes the inner product of vectors x, y. If $A \in \mathbb{R}^{d\times r}$, then its trace norm is defined as $\|A\| := \sqrt{\text{trace}(A^{\mathsf{T}}A)}$. For arbitrary $a, b \in \mathbb{R}$, we denote $\max(a, b)$ and $\min(a, b)$ by $a \vee b$ and $a \wedge b$, respectively. We define $\inf \emptyset = \infty$. Let $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}_+)$ denote the family of all real-valued functions $V(t, x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^d$ such that they are continuously twice differentiable in x and once in t. $\{t\}$ denotes the fractional part of t.