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The halfspace matching method: A new method to solve scattering problems in infinite media

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ABSTRACT

We are interested in acoustic wave propagation in time harmonic regime in a two-dimensional medium which is a local perturbation of an infinite isotropic or anisotropic homogeneous medium. We investigate the question of finding artificial boundary conditions to reduce the numerical computations to a neighborhood of this perturbation. Our objective is to derive a method which can extend to the anisotropic elastic problem for which classical approaches fail. The idea consists in coupling several semi-analytical representations of the solution in halfspaces surrounding the defect with a Finite Element computation of the solution around the defect. As representations of the same function, they have to match in the infinite intersections of the halfspaces. It leads to a formulation which couples, via integral operators, the solution in a bounded domain including the defect and its traces on the edge of the halfspaces. A stability property is shown for this new formulation.

1. Introduction and model problem

This work is motivated by the numerical simulation of Non Destructive Testing or Structural Health Monitoring experiments in anisotropic elastic media (see for instance [1]). More precisely, we are interested in simulating the diffraction of time-harmonic waves by a localized perturbation in a homogeneous two dimensional infinite anisotropic elastic medium. Since the medium is infinite, there are theoretical difficulties – how to define the so called outgoing solution of such problem? – and numerical difficulties — can we introduce an equivalent formulation which is suitable for numerical purposes (for instance a formulation set in a bounded domain with appropriate boundary conditions)?

This is an old problematic [2] for time harmonic scalar wave equations and there exist several methods. They are all based on the natural idea of reducing the pure numerical computations to a bounded domain containing the perturbations (achieved using for instance Finite Element methods). A first class of methods consists in applying an artificial boundary conditions, around the bounded domain, which is transparent or approximately transparent as in: (1) integral equation techniques, (2) Dirichlet-to-Neumann approaches providing that the boundary is properly chosen to allow separation of variables and (3) local radiation conditions at finite distance constructed as local approximations at various order of the exact non local condition. These techniques were first introduced for the time harmonic scalar wave equation – the Helmholtz equation – and then extended to isotropic elasticity problems using simply the Helmholtz decomposition of the displacement field in terms of potential (see for instance [3]). However it seems that all these techniques either do not extend to anisotropic elastic media – the separation of variables is not possible anymore to determine the Dirichlet-to-Neumann (DtN) operator –

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or do extend but with a tremendous computational cost — for the integral equation techniques, the Green tensor depends not only on the distance between two points but also on the orientation [4]. A second class of methods consists in surrounding the computational domain by a Perfectly Matched absorbing Layer (PML). PML techniques are very popular because they are efficient and easy to implement in a large class of problems. But they may be inoperant. Roughly speaking, the PML absorbs the wave with an outgoing phase velocity, preventing them to come back in the computational domain, while in order to catch the physical solution, it should absorb the waves with outgoing group velocities. That is why to our knowledge the standard PML technique works for isotropic elastic media (in which the waves with outgoing phase velocities have outgoing group velocities and vice versa) but cannot work for general anisotropic elastic media where the two velocities may differ [5–7]. Let us mention finally the pole condition method which was developed recently and adapted to anisotropic elastic waveguide in the past few years [8,9].

By contrast, our method which is inspired by the method developed in [10,11] for locally perturbed periodic media can cover all the cases. It is based on a simple idea: the solution of homogeneous – isotropic or anisotropic, acoustic or elastic – halfspace problems can be expressed thanks to its trace on the halfspace boundary. As several halfspaces surrounding the perturbations are needed to recover the whole domain, they will necessarily overlap. The second step is then to find conditions to ensure the compatibility of the representations in the overlapping zones. This method has links with domain decomposition methods with overlap [12–14], with the specific difficulty that here the overlapping zones are unbounded. More precisely, the idea is to split the whole domain into five parts:

- a square that includes the defect (and all the inhomogeneities) in which we will use a Finite Element representation of the solution,
- and 4 half-planes, parallel to the four edges of the square in which the medium is homogeneous.

Taking advantage of the homogeneity of the medium in a half-plane, we can give an explicit (integral) expression of the solution given (for instance) its trace on the edge of the half-plane, via the Fourier transform in the *transverse direction*. With these integral representations and the Finite Element representation of the solution in the square, we can formulate a coupled problem. To ensure the compatibility of the different representations, as in domain decomposition methods, we impose transmission conditions on the edges of the subdomains. This leads us to a system of coupled equations where the unknowns are the solution in the bounded square and the traces of the solution on the edges of the half-planes.

Obviously, compared to absorbing layers methods, this approach is more costly due to the additional unknowns (the traces) linked by non-local integral equations. One counterpart is that this additional computation of the traces enables to reconstruct a posteriori the solution in the half-planes (and therefore in the whole domain), which is impossible for instance when using non exact absorbing boundary conditions or PML.

In the paper, we consider the simple model case of a scalar equation: the dissipative anisotropic Helmholtz equation. We reformulate the diffraction problem and analyze the properties of this reformulation. Let us underline that, though our analysis holds only for dissipative media, the method gives good numerical results also in the non-dissipative case. We will explain, in a dedicated section, the theoretical difficulties raised by the case without dissipation. We should also emphasize that this approach remains valid for anisotropic elastic media since it mainly relies on the homogeneity of the medium in the half-planes to get the Fourier representations (see [15]). *The complete description of the method in the elastic case will be the topic of another paper.*

The general model problem that we consider in this paper is then

$$\left| \; -\mathrm{div}(A(x,y)\nabla p) - \omega_\varepsilon^2\, \rho(x,y)p = f \;\text{ in }\; \Omega, \right. \tag{1}$$

in the time harmonic regime at the frequency $\mathrm{Re}(\omega_\varepsilon) = \omega$ with a small absorption $\mathrm{Im}(\omega_\varepsilon) = \varepsilon > 0$, where A is a symmetric positive definite matrix of $(L^\infty(\Omega))^{2\times 2}$ modeling the anisotropy and ρ is a strictly positive function of $L^\infty(\Omega)$.

The propagation domain Ω is typically $\mathbb{R}^2$, or $\mathbb{R}^2$ minus a set of obstacles which are included in a bounded region

$$\exists a > 0, \quad \partial\Omega \subset \Omega_a \equiv (-a, a)^2.$$

In presence of obstacles, some boundary conditions have to be added to the model. The source term f is supposed to be a function of $L^2(\Omega)$ with a compact support included in Ω_a. Finally, the matrix A is a local perturbation of a constant matrix A_0

$$\mathrm{supp}(A - A_0) \subset \Omega_a, \text{ where } A_0 = \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} \text{ with } \begin{cases} c_1, c_2 > 0, \\ c_1c_2 - (c_3)^2 > 0, \end{cases} \tag{2}$$

and the function ρ is a local perturbation of a constant function, which is taken, without loss of generalities, equal to 1

$$\mathrm{supp}(\rho - 1) \subset \Omega_a. \tag{3}$$

For variational boundary conditions on $\partial\Omega$ – for instance Neumann or Dirichlet conditions – it is well known that thanks to the dissipation, this problem admits a unique solution in $H^1(\Omega)$.

To clarify the presentation of the method, we will consider three situations of increasing difficulty.