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Limit cycles of continuous and discontinuous piecewise-linear differential systems in $\mathbb{R}^3$

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ABSTRACT

We study the limit cycles of two families of piecewise-linear differential systems in $\mathbb{R}^3$ with two pieces separated by a plane Σ. In one family the differential systems are continuous on the plane Σ, and in the other family they are discontinuous on the plane Σ.

The usual tool for studying these limit cycles is the Poincaré map, but here we shall use recent results which extend the averaging theory to continuous and discontinuous differential systems.

All the computations have been done with the algebraic manipulator Mathematica.

1. Introduction and statement of the main results

The study of piecewise linear differential systems essentially started with Andronov, Vitt and Khaikin [1] and still continues to receive attention by researchers. The continuous and discontinuous piecewise-linear differential systems play an important role inside the nonlinear dynamical systems. First they appear in a natural way in nonlinear engineering models, where certain devices are accurately modeled by such differential systems, see for instance the books of di Bernardo, Budd, Champneys and Kowalczyk [2], and Simpson [3], the survey of Makarenkov and Lamb [4], and the hundreds of references quoted in these last three works. Moreover these kinds of differential systems are frequent in applications from electronic engineering and nonlinear control systems, where they cannot be considered as idealized models; they are also used in mathematical biology as well, see for instance [5–8].

There are many studies of the limit cycles of continuous and discontinuous piecewise-linear differential systems in $\mathbb{R}^2$ with two pieces separated by a straight line, see for instance [9–27]. But there are few results about the limit cycles of continuous and discontinuous piecewise-linear differential systems in $\mathbb{R}^3$ with two pieces separated by one plane, see for example [28–30]. The objective of this work is to study the limit cycles of some of these last systems.

We consider perturbations of the linear differential system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x, \\ \dot{z} &= sx, \end{aligned} \tag{1}$$

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with $(x, y, z) \in \mathbb{R}^3$ and s a real parameter. The dot denotes derivative with respect to an independent variable t, usually called the time. Straightforward computations show that the solutions of system (1) are all periodic with the exception of the z-axis which is filled with equilibria.

In this paper first we study the periodic solution of the following perturbed continuous piecewise linear differential system

$$\begin{aligned} \dot{x} &= -y + \varepsilon(a + bx + cy + d|z|), \\ \dot{y} &= x + \varepsilon(e + fx + gy + h|z|), \\ \dot{z} &= sx + \varepsilon(j + kx + ly + m|z|), \end{aligned} \tag{2}$$

of system (1) with two zones $z > 0$ and $z < 0$, where $a, b, c, d, e, f, g, h, j, k, l$ and m are real parameters and the parameter $\varepsilon > 0$ is sufficiently small. Changing z by $-z$ if necessary, we always can assume that the parameter $s \geq 0$. The reason of perturbing the linear differential system (1) is due to the fact that this is the linear differential system in $\mathbb{R}^3$ having more periodic solutions, because all its solutions are periodic except the z-axis which is filled of singular points.

Our main result on the periodic solutions of the continuous piecewise linear differential system (2) is the following. This result is obtained using the extension of the classical averaging theory for smooth differential systems to continuous differential systems given in [31], see Section 2 for more details.

Theorem 1. *For the continuous piecewise differential system (2) the following statements hold.*

(a) *For $\varepsilon > 0$ sufficiently small if $s > 0$ there exist values r^* and z^* such that system (2) has the periodic solution*

$$(x(t), y(t), z(t)) = (r^* \cos t + O(\varepsilon), r^* \sin t + O(\varepsilon), z^* + sr^* \sin t + O(\varepsilon)),$$

if

$$\frac{(b+g)}{hs} \in [-1, 0) \cup (0, 1] \quad \text{and} \quad \frac{(es-j)}{s(m-hs)} > 0.$$

(b) *If $s = 0$ system (2) has two periodic solutions*

$$(x(t), y(t), z(t)) = (\mu_1 \cos t + O(\varepsilon^2), \mu_2 \sin t + O(\varepsilon^2), \pm j/m + O(\varepsilon^2)),$$

where $\mu_1 = O(\varepsilon)$ and $\mu_2 = O(\varepsilon)$, if $(b+g)m \neq 0$, $j/m < 0$ and $\varepsilon > 0$ sufficiently small is such that $O(\varepsilon) > 0$.

Theorem 1 is proved in Section 3 using the averaging theory of first order for the continuous piecewise linear differential systems. We note that the averaging theory is one of the few tools that when it can be applied allows to prove analytically the existence of periodic orbits. We remark that in general this tool does not find all the periodic orbits of a differential system. Another good tool for studying analytically the periodic solutions of non-smooth differential systems is the Melnikov theory. Thus see for instance the papers [32,33] where the authors studied planar non-smooth systems using the Melnikov theory, and also the papers [34,35] for 3-dimensional non-smooth systems. In fact Melnikov theory and averaging theory are essentially different formulation of equivalent theories, see for details [36].

Many problems in physics, economics, biology and applied areas are modeled by discontinuous differential systems but there exist only few analytical techniques for studying their periodic solutions. In [37] the authors extended the averaging theory to discontinuous differential systems. An improvement of this result for a much bigger class of discontinuous differential systems is given in [38].

Applying these tools we also investigate the periodic solutions of the discontinuous piecewise linear differential system

$$\begin{aligned} \dot{x} &= -y + \varepsilon(a + bx + cy + df(z)), \\ \dot{y} &= x + \varepsilon(e + fx + gy + hf(z)), \\ \dot{z} &= sx + \varepsilon(j + kx + ly + mf(z)), \end{aligned} \tag{3}$$

with two pieces defined by $f(z) = z + \text{sign}(z)$ and

$$\text{sign}(z) = \begin{cases} 1 & \text{if} \quad z > 0, \\ -1 & \text{if} \quad z < 0. \end{cases}$$

We get the following result on the periodic solutions of the discontinuous piecewise linear differential system (3).

Theorem 2. *Using the averaging theory of first order for the discontinuous piecewise linear differential system (3), the following statements hold.*

(a) *If $s > 0$ and $\left|\frac{es-j}{hs-m}\right| \in (0, 1]$, for $\varepsilon > 0$ sufficiently small there exist values r^* such that system (3) has the crossing periodic solution*

$$(x(t), y(t), z(t)) = (r^* \cos t + O(\varepsilon), r^* \sin t + O(\varepsilon), z^* + sr^* \sin t + O(\varepsilon)),$$

where $z^* = -\frac{sr^*}{4}\sqrt{16 - \frac{\pi^2(b+g+hs)^2(r^*)^2}{h^2}}$ if $\frac{es-j}{hs-m} \in (0, 1]$ and $z^* = \frac{sr^*}{4}\sqrt{16 - \frac{\pi^2(b+g+hs)^2(r^*)^2}{h^2}}$ if $-\frac{es-j}{hs-m} \in (0, 1]$.