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Realizable algorithm for approximating Hilbert–Schmidt operators via Gabor multipliers

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ABSTRACT

In this work, we consider new computational aspects to improve the approximation of Hilbert–Schmidt operators via generalized Gabor multipliers. One aspect is to consider the approximation of the symbol of an Hilbert–Schmidt operator as L^2 projection in the spline-type space associated to a Gabor multiplier. This gives the possibility to employ a selection procedure of the analysis and synthesis function, interpreted as time-frequency lag; hence, with the related algorithm it is possible to handle both underspread and overspread operators. In the numerical section, we exploit the case of approximating overspread operators having compact and smooth spreading function and discontinuous time-varying systems. For the latter, the approximation of discontinuities in the symbol is not directly achievable in the generalized Gabor multipliers setting. For this reason, another aspect is to further process the symbol through a Hough transform, to detect discontinuities and to smooth them using a new class of approximants. This procedure creates a bridge between features detection techniques and harmonic analysis methods, and in specific cases it almost doubles the accuracy of approximation.

1. Introduction

The problem of approximating Hilbert–Schmidt (HS) operators via Gabor multipliers was approached by several authors [1–3] due to its important applications in computational sciences and signal processing [4]. In [1], the authors have improved the approximation of HS operators using Generalized Gabor Multipliers (GGM). In [2], the authors present the connection between the approximation of the HS operators and GGM via multi-window spline-type spaces. The connection was presented only theoretically in that paper and no implementation was given. In this work, we have started from the theory proposed in [2] and we have designed a realizable algorithm that improves the overall numerical approximation of the HS operators in three ways: in speed by transferring the problems in splines-spaces domain, in complexity by lowering the number of atoms through a proper selection procedure, and geometrically via a feature detection technique i.e. the Hough transform. In a previous work [5], we have used the reformulation of Gabor system in spline-type spaces to produce a sparse-frequency representation of a signal. We use in this paper a generalized Gabor multipliers for the same purpose: selection of proper *channels* for the approximation of overspread or underspread operators. To do this, we will not use Euclidean spaces and we prefer an approach that uses locally compact (LC) groups as done in [6]. Also, we prefer a more classical approach [7] that does not involve modulation spaces [8] since we restrict our study to the class of HS operators. This general theory will lead to usual L^2-projection once restricted to square integrable functions. The channels will not be

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orthogonal, but biorthogonal. For the class of HS operators we will obtain L^2-projection, hence it can be seen as optimal design of biorthogonal frequency division multiplex (BFDM) [9] in a Gaborian dictionary.

In Section 2 we outline a minimal necessary theory for introducing the novel concepts. In Section 3 we introduce GGM seen as elements of a spline-type (ST) space. In Section 3.1 we exploit the connection between GGM and ideals of the related ST spaces. This gives us a novel procedure for the selection of analysis and synthesis functions of a GGM. In Section 4 we will test the algorithm and analyse the outcome. The outcome is similar to the one developed in [10] but it handles the multi-window formulation of GGM. We show the robustness of the generators selection procedure for operators having smooth symbol and the inability of GGM of handling discontinuities. We overcome this problem by looking at the reminder as an image and using a common tool in image processing and pattern recognition: the Hough transform (HG) [11].

2. Theoretical brief

Given a LC group G the convolution for the space $L^1(G)$ can be defined as $f * g := \int_G f(x)L_x g(y)dx$, where $L_y f(x) := f(y^{-1}x)$ is the left shift. The morphisms from G into the torus form a LC group $\widehat{G}$ called the dual group and each element, called character, can be represented as $x \longmapsto \langle x, \hat{x} \rangle$ $x \in G$. The multiplication of a function with a character is called modulation, as in the case of the real groups $\mathbb{R}^d$ having as characters the Fourier basis $\langle x, \hat{x} \rangle := e^{2i\pi x \cdot \hat{x}}$, $x, \hat{x} \in \mathbb{R}^d$. The Fourier transform of a function in $L^1(G)$ is defined as $\mathcal{F}(f)(\hat{x}) := \int_G f(x)\overline{\langle x, \hat{x} \rangle}dx$ $\hat{x} \in \widehat{G}$ which composes with shift and character multiplication as

$$\mathcal{F}(L_y f)(\hat{x}) = \overline{\langle y, \hat{x} \rangle}\mathcal{F}(f)(\hat{x}) \tag{1}$$

$$\mathcal{F}(\langle \cdot, (\hat{y}) \rangle f)(\hat{x}) = L_{\hat{y}}\mathcal{F}(f)(\hat{x}). \tag{2}$$

The "hat" notation is used in the mathematical literature for both the Fourier transform of function as well as for members of the dual group: $\mathcal{F}(f)(\hat{x})$ or alternately $\widehat{f}(\widehat{x})$. Given $\sigma \in S'(\mathbb{R}^{G \times \widehat{G}})$, a tempered distribution, we can define the operator $PDO_\sigma : S(G) \to S'(G)$ acting on the Schwartz class $S(G)$ as

$$PDO_\sigma f(x) := \int_{\widehat{G}} \sigma(x, \widehat{x}) \left(\int_G f(t)\overline{\langle t, \widehat{x} \rangle} dt \right) \langle x, \widehat{x} \rangle \, d\widehat{x}. \tag{3}$$

It is in fact an application of the Fourier transform to f, followed by multiplication with the mask σ, also called the Kohn–Nirenberg (K–N) symbol of the operator, and finally an application of the inverse Fourier transform. The K–N quantization rule establishes a correspondence between $L^2(G \times \widehat{G})$ and HS operator defined on $L^2(G)$ [12]. Another important element to analyse such an operator is its spreading function, which is the Fourier transform of its K–N symbol.The spreading function was mainly used in pseudodifferential operators theory [13,14] for the possibility of expressing an operator PDO_σ as $PDO_\sigma f = \int_{\widehat{G}} \int_G \widehat{\sigma}(\widehat{x}, x) \langle \cdot, \widehat{x} \rangle L_{x^{-1}} f d\widehat{x}\, dx$. Our use of spreading function is instead closer to classification of operators. A PD (pseudodifferential) operator having spreading function well localized in 0 is called underspread, otherwise it is called overspread. The same distinction is made in the study of nonstationary random processes [15,16]. Overspread processes are related to high-lag time-frequency correlation; in Section 3.1 we will find the same lag behaviour analysing GGM through their spreading functions.

3. Gabor multipliers as Rihaczek spline-type systems

We want to produce an expansion of the symbol σ through well localized *blocks* in the phase space, as it happens for the approximation of a function through Gabor systems [17]. Given $\boldsymbol{g} := \{g_i : i = 1, \dots, r\}$ and $\boldsymbol{\gamma} := \{\gamma_i : i = 1, \dots, r\}$, two sets of tempered distributions over G, and H a lattice in $G \times \widehat{G}$, a Generalized Gabor multiplier (GGM for short) is an operator in the form

$$M_{g,\gamma,H,m} f := \sum_{i=1}^{r} \sum_{\lambda \in H} m_i(\lambda) \langle f, \pi(\lambda) g_i \rangle \, \pi(\lambda)\gamma_i \tag{4}$$

where m_i is the mask and the left shift operator and modulation operator, usually denoted as L_y and $M_{\widehat{x}}$ [18], are joined in an unique time-frequency shift $\pi(x, \widehat{x}) := M_{\widehat{x}} L_y$. If the analysis and synthesis sets are composed by only one function the operator is called Gabor Multiplier (GM).

The rank one operator $Q_{g,\gamma} f := \langle f, g \rangle \, \gamma$ can be seen as a PD operator having the symbol $\sigma_{g,\gamma}(x, \widehat{x}) = g(x)\overline{\widehat{\gamma}(\widehat{x})}\, \overline{\langle x, \widehat{x} \rangle}$, called Rihaczek distribution of g and γ.

The central property of this operator is that $\sigma_{\pi(\lambda)g, \pi(\lambda)\gamma} = L_\lambda^{G \times \widehat{G}} \sigma_{g,\gamma}$, with $L^{G \times \widehat{G}}$ being the shift that acts componentwisely [6], hence it is possible to avoid the inversion of the twisted convolution [19] that traditionally occurs by analysing the symplectic nature of the phase space [18]. By linearity of quantization, the approximation of a PD operator in the class $\mathbb{M}_{g,\gamma,H}$ of GGM generated by (g, γ, H) is a spline-type (ST) approximation of its symbol [20].

Definition 1. Let G be a LC group, H be a closed and normal subgroup of G, $\mathbb{B}$ be a Banach space, and $\Phi = \{\phi_i : G \to \mathbb{B}\}_{i \in I}$ be any set of functions or distributions in a translation invariant Banach space $(\mathcal{B}, \| \cdot \|_B)$. We call *spline-type system* generated by Φ and H the collection of left shifts $\{L_a \phi_i : a \in H, \ i \in I\}$, and *spline-type space* $\mathcal{S}(\Phi, H)$ its closed span in $\mathcal{B}$.