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The block CMRH method for solving nonsymmetric linear systems with multiple right-hand sides

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ABSTRACT

CMRH method (Changing minimal residual with Hessenberg process) is an iterative method for solving nonsymmetric linear systems. This method is similar to QMR method but based on the Hessenberg process instead of the Lanczos process. On dense matrices, the CMRH method is less expensive and requires less storage than other Krylov methods. This paper presents a block version of the CMRH algorithm for solving linear systems with multiple right-hand sides. The new algorithm is based on the block Hessenberg process and the iterates are characterized by a block version of the quasi-minimization property. We analyze its main properties and show that under the condition of full rank of block residual the block CMRH method cannot break down. Finally, some numerical examples are presented to show the efficiency of the new method in comparison with the traditional CMRH method and a comparison with the block GMRES method is also provided.

1. Introduction

We consider the solution of large linear systems with multiple right-hand sides of the form

$$AX = B, \tag{1}$$

where A is an $n \times n$ nonsingular matrix, and $B = [b^{(1)}, \ldots, b^{(s)}], X = [x^{(1)}, \ldots, x^{(s)}]$ are rectangular $n \times s$ matrices. In practice, s is small relative to n. Many applications such as in electromagnetic scattering problem and in structural mechanics problems require the solution of linear systems with multiple right-hand sides (1).

For large problems, many iterative methods, which are the generalizations of the classical Krylov subspace methods, have been proposed in recent years. One class of solvers is the global Krylov subspace methods, which are based on the use of a global projection process onto a matrix Krylov subspace. References on this class include [1–11]. Another class is the seed methods, which consist of selecting a single system as the seed system and generating the corresponding Krylov subspace and then projecting all the residuals of the other linear systems onto the same Krylov subspace to find new approximate solutions as initial approximations. References on this class include [12–14] . The other class is the block Krylov subspace solvers which are more suitable for dense systems with preconditioner. The Block Conjugate Gradient (BCG) is the first block iterative solver introduced by O'Leary [15], its related algorithms were proposed for parallel computers [16], and a breakdown-free block conjugate gradient method presented in [17]. For nonsymmetric problems, the block generalized minimal residual (Bl-GMRES) algorithm [18–23], the block quasi minimum residual (Bl-QMR) algorithm [24], the block

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BiCGStab (Bl-BICGSTAB) algorithm [25,26], the block Lanczos method [27] , the block least squares (Bl-LSQR) algorithm [28], the block IDR(s) method [29], the block GCROT(m, k) method [30], and the block LSMR method [31], have been developed.

In this paper, we generalize the CMRH method [32] to solve linear systems with multiple right-hand sides. The proposed method is referred to as block CMRH (Bl-CMRH) method. We analyze its main properties and show that under the condition of full rank of block residual the block CMRH method cannot break down.

The paper is organized as follows. In Section 2, we shortly review the Hessenberg process and CMRH method. We present the block version of the Hessenberg process and its properties in Section 3. In Section 4, we describe the block CMRH method. Section 5 demonstrates the effectiveness of the proposed method. Conclusions are summarized in Section 6.

Throughout the paper, all vectors and matrices are assumed to be real. For a vector v, $\|v\|$ always denotes the Euclidean norm $\|v\| = \sqrt{(v^T v)}$ and $\|v\|_\infty$ denotes the maximum norm $\|v\|_\infty = \max_{i=1,\dots,n}|v_i|$, where v_i is the ith component of the vector v. For a matrix X, $\|X\|_F$ denotes the Frobenius norm $\|X\|_F = \sqrt{\mathrm{tr}(X^T X)}$. For a matrix $V \in \mathbb{R}^{n\times s}$, the block Krylov subspace $\mathcal{K}_k(A, V)$ is the subspace generated by the columns of the matrices $V, AV, A^2V, \dots, A^{k-1}V$. Some MATLAB notation is used; for instance, $H_k(i+1:m+1, 1:m)$ denotes the portion of H_k with rows from $i+1$ to $m+1$ and columns from 1 to m. Finally, 0_s and I_s will denote the zero and the identity matrices in $\mathbb{R}^{s\times s}$, respectively.

2. CMRH method

In this section, we consider the linear system of equations

$$Ax = b, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsingular and nonsymmetric matrix, $b \in \mathbb{R}^n$ is a given vector.

The CMRH method is an algorithm for solving nonsymmetric linear systems in which the Arnoldi component of GMRES is replaced by the Hessenberg process, which generates Krylov basis vectors which are orthogonal to standard unit basis vectors rather than mutually orthogonal. The iterate is formed from these vectors by solving a small least squares problem involving a Hessenberg matrix.

Let v be a column vector of $\mathbb{R}^n$ and A an $n \times n$ real matrix. The Hessenberg reduction process (without pivoting strategy) computes a unit trapezoidal matrix $L_m = [l_1, \dots, l_m]$ whose columns form a basis of the Krylov subspace $\mathcal{K}_m(A, v) = \mathrm{span}\{v, Av, \dots, A^{k-1}v\}$ by using the following formulas:

$$\begin{cases} \beta = (v)_1, & l_1 = v/\beta, \\ h_{k+1,k} l_{k+1} = Al_k - \sum_{j=1}^{k} h_{j,k} l_j, & \text{for } k = 1, \dots, m. \end{cases}$$

The parameters $h_{j,k}$ are determined such that

$$l_{k+1} \perp e_1, \dots, e_k \text{ and } (l_{k+1})_{k+1} = 1.$$

The Hessenberg process can break down if $h_{k+1,k}$ is zero. We can avoid such a breakdown and also ensure numerical stability if we use a pivoting strategy such as in the Gaussian elimination method (see, for example, [33]). Algorithm 1 summarizes the Hessenberg process with pivoting strategy [34].

Algorithm 1 Hessenberg process with pivoting strategy

1. $p = [1, 2, \dots, n]^T$,
   Determine i_0 such that $|(v)_{i_0}| = \|v\|_\infty$, $\beta = (v)_{i_0}$, $l_1 = \dfrac{v}{\beta}$, $p_1 \leftrightarrow p_{i_0}$
2. For $k = 1, \dots, m$
   $u = Al_k$,
   For $j = 1, \dots, k$
   $h_{j,k} = (u)_{p_j}$,
   $u = u - h_{j,k} l_j$,
   end
   If ($k < n$ and $u \neq 0$) then
   Determine $i_0 \in \{k+1, \dots, n\}$ such that $|(u)_{p_{i_0}}| = \|(u)_{p_{k+1:p_n}}\|_\infty$,
   $h_{k+1,k} = (u)_{p_{i_0}}$, $l_{k+1} = u/h_{k+1,k}$, $p_{k+1} \leftrightarrow p_{i_0}$,
   else
   $h_{k+1,k} = 0$, stop.
   end
   end