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Some results on generalized Szász operators involving Sheffer polynomials

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ABSTRACT

In this paper we consider old and new operators of Szász type involving Sheffer polynomials. We present an asymptotic expansion formula for operators of Ismail type. Then, in order to improve the accuracy of the approximation of a function f in a fixed point, we apply a well-known extrapolation algorithm. We also introduce some new special sequences of Appell and Sheffer polynomials and construct new generalized Szász-type operators. By using classical techniques we investigate approximation properties and rate of convergence for these operators and compare the results with other existing operators. Finally, we present numerical examples which confirm the validity of the theoretical analysis and the effectiveness of the presented operators.

1. Introduction

In the approximation theory one of the fundamental problems is to approximate a function f by functions having better properties of integration, differentiation and efficient calculations than f. After the famous Theorems of Weierstrass and Korovkin [1], positive linear polynomial operators have been widely used for approximating regular functions.

In this paper we are dealing with the approximation of real functions f defined in the semi-infinite interval $[0, +\infty)$ which have a suitable rate of growth as $x \to \infty$. In this case, one of the well-known examples of sequences of positive linear operators is Szász operators, introduced by Szász in 1950 [2]:

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{1}$$

S_n converges to f (as $n \to \infty$) at each point $t = x \geq 0$ where f is continuous. In [2] Szász investigated the detailed approximation properties of the operators (1). Particularly, if f is bounded in every finite interval and $f(x) = O\left(x^k\right)$ for some $k > 0$ as $x \to \infty$, then there hold:

– if f is continuous at a point ξ, then S_n converges uniformly to f at ξ;

– if f is differentiable at a point $\xi > 0$, then $n^{\frac{1}{2}} \left[S_n(f;\xi) - f(\xi)\right] \to 0, n \to \infty$.

Later, in 1969, Jakimovski and Leviatan [3] gave a generalization of Szász operators by means of Appell polynomials:

$$P_n(f;x) = \frac{e^{-nx}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{2}$$

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where $\{p_k\}_{k\in\mathbb{N}}$ is the Appell polynomial sequence [4] defined from the holomorphic function $A(t)$ in the disk $|z| < R$, $(R > 1)$ with $A(1) \neq 0$.

Under the assumption $p_k(x) \geq 0$, for $x \in [0, \infty)$ and $k \geq 0$, for each function f defined in $[0, \infty)$, Jakimovski and Leviatan [3] gave several approximation properties of these operators in view of Szász's method [2].

If $A(t) = 1$, $p_k(x) = \frac{x^k}{k!}$, then $P_n(f; x)$ reduces to the Szász operators (1).

Let $E = E[0, \infty)$ denote the class of all functions of exponential type defined on the semi-axis and such that $|f(t)| < ce^{\alpha t}$ $(t \geq 0)$ for some c and α finite positive constants and let $C_E[0, \infty) = C[0, \infty) \cap E$. Observe that, if $n > \alpha \log R$, the series in (2) is convergent. Hence the operators $P_n(f; x)$ are well-defined for all sufficiently great n.

In [3] the authors proved that, if $f \in C_E[0, \infty)$, then $P_n(f; x)$ converges to $f(x)$ as $n \to \infty$. The convergence is uniform in each compact subset $[0, a]$, $a > 0$.

In 1996 Ciupa [5] studied the rate of convergence of these operators.

If f admits derivatives of sufficiently high order at $x \geq 0$, a complete asymptotic expansion for $P_n(f; x)$ has been derived by Abel and Ivan [6]:

$$P_n(f; x) \sim f(x) + \sum_{k=1}^{\infty} c_k(f; x) n^{-k}, \tag{3}$$

where the coefficients $c_k(f; x)$ do not depend on n. From (3) it follows that for all $m \geq 1$

$$P_n(f; x) = f(x) + \sum_{k=1}^{m} h^k c_k(f; x) + o\left(h^m\right), \tag{4}$$

with $h = 1/n$.

Operators of Jakimovski and Leviatan type have been generalized by substituting the Appell sequence by a more general Sheffer sequence [7]. In particular, in [8] Ismail considered the polynomial operators

$$F_n(f; x) = \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{5}$$

where $\{p_k\}_{k\in\mathbb{N}}$ is the Sheffer polynomial sequence related to the analytic functions $A(t)$ and $H(t)$ with $A(0) \neq 0$, $H(0) = 0$ and $H'(0) \neq 0$. Moreover, in (5), $A(1) \neq 0$, $H(1) \neq 0$. Under the assumptions $H'(1) = 1$, $p_k(x) \geq 0$ for $x \geq 0$, Ismail showed that the same type of results obtained by Jakimovski and Leviatan are still valid for operators $F_n(f; x)$.

Observe that for $H(t) = t$, operators $F_n(f; x)$ reduce to operators $P_n(f; x)$.

Further generalizations of Szász-type operators have been studied in [9–12].

To the knowledge of the authors, an explicit asymptotic expansion for operators (5) does not exist in the literature. Therefore, in this paper, with the same technique used in [6], a complete asymptotic expansion formula for the Ismail-type operators $F_n(f; x)$ is given, provided f admits derivatives of sufficiently high order at $x \geq 0$. Asymptotic expansions of type (4) are very important, since a well-known extrapolation algorithm can be applied, in order to improve the approximation results. This algorithm provides new sequences of operators having a faster rate of convergence than $F_n(f; x)$.

The paper is structured as follows. In Section 2 necessary background and definitions are given; in Section 3 the asymptotic expansion for $F_n(f; x)$ is presented and Richardson extrapolation technique is applied for any fixed $x \in [0, +\infty)$. Then, in Section 4 some known and new special operators of Jakimovski and Leviatan type and of Ismail type are considered. Rates of convergence of the above operators are compared by using classical techniques. Numerical experiments are given in Section 5 which provide favorable comparisons with other existing operators and show the effectiveness of the considered extrapolation algorithm. Finally, conclusions are reported in Section 6.

2. Preliminaries

In this section we recall some well-known definitions and properties on Appell and Sheffer polynomials which will be useful hereafter. Moreover the estimation of the rate of convergence for $P_n(f, x)$ and $F_n(f, x)$ operators by means of modulus of continuity is examined.

Appell sequences [4,13–15] and Sheffer sequences [7,13,15–17] can be introduced in many equivalent ways. Here we use the method of generating functions.

If $A(t)$ is an invertible power series, the polynomial sequence $\{p_k\}_{k\in\mathbb{N}}$ defined by

$$A(t)e^{xt} = \sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!} \tag{6}$$

is called *Appell polynomial sequence* for $A(t)$. If

$$A(z) = \sum_{i=0}^{\infty} a_i \frac{z^i}{i!}, \tag{7}$$