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Convergence rate and stability of the truncated Euler–Maruyama method for stochastic differential equations

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ABSTRACT

Recently, Mao (2015) developed a new explicit method, called the truncated Euler–Maruyama (EM) method, for the nonlinear SDE and established the strong convergence theory under the local Lipschitz condition plus the Khasminskii-type condition. In his another follow-up paper (Mao, 2016), he discussed the rates of L^q-convergence of the truncated EM method for $q \geq 2$ and showed that the order of L^q-convergence can be arbitrarily close to $q/2$ under some additional conditions. However, there are some restrictions on the truncation functions and these restrictions sometimes might force the step size to be so small that the truncated EM method would be inapplicable. The key aim of this paper is to establish the convergence rate without these restrictions. The other aim is to study the stability of the truncated EM method. The advantages of our new results will be highlighted by the comparisons with the results in Mao (2015, 2016) as well as others on the tamed EM and implicit methods.

1. Introduction

Influenced by Higham, Mao and Stuart [1], the strong convergence theory of numerical methods for nonlinear stochastic differential equations (SDEs) without the global Lipschitz condition has become more and more popular. Although the classical Euler–Maruyama (EM) method is convenient for computations and implementations, the absolute moments of its approximation for SDEs with super-linear coefficients diverge to infinite at a finite time (see, e.g., [2]). Many implicit methods were used to study the numerical solutions to SDEs with nonlinear coefficients (see, e.g., [1,3–7]). Especially, Higham, Mao and Stuart [1] proved that the implicit EM numerical solutions converge strongly to the exact solutions of SDEs with globally one-sided Lipschitz continuous drift term and globally Lipschitz diffusion term, but the explicit EM method fails to do that. For the background on the implicit methods, we refer the reader to the books [8–10]. However, it is demonstrated that the implementation of the implicit EM method requires more computational effort. Recently, due to the advantages of explicit methods, Hutzenthaler, Jentzen and Kloeden proposed an explicit method for such SDEs called tamed Euler method whose numerical solutions converge strongly to the exact solution with 1/2 order. Sabanis in [11] went a further step to propose the modified tamed Euler method approximating the SDEs with superlinearly growing drift and diffusion coefficients, moreover, recovered the strong order 1/2 in the estimation of convergence rate. Other explicit methods, such as the stopped EM method, as well as the tamed Milstein method, have been further developed (see, e.g., [12,13] for details).

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In particular, Mao [14] in 2015 proposed a new explicit method, called the truncated EM method. In his another follow-up paper [15], he investigated the convergence rates for the method under some additional conditions. We will point out that some of these additional conditions might force the step size to be so small that the truncated EM method would be inapplicable. One of our key aims in this paper is to establish the convergence rate without these restrictions so that the truncated EM method is more widely implementable. To overcome the difficulties due to removing these restrictions, some new mathematical techniques, which are significantly different from those used in [15], have been developed.

A nice numerical method should not only have an acceptable finite-time convergence rate but also have the ability to preserve the asymptotic properties of the underlying SDEs (see, e.g., [16,17]). Another aim of this paper is to show the ability of the truncated EM method to preserve the asymptotic stability of the underlying SDEs.

To show the advantages of the truncated EM method, we will compare it with other methods, e.g., the implicit EM method, the tamed Euler method and the modified tamed Euler method. We will design two numerical experiments and compute the errors between the true solution and the numerical solutions obtained by different schemes. It turns out that to achieve the same accuracy, the runtime of the truncated EM method and of the tamed Euler method are almost equivalent, but much shorter than that of the implicit EM method. However, to achieve the same accuracy, the step size for the modified tame Euler method is required to be smaller than that for the truncated EM method. These show clearly that the truncated EM method might be more efficient and is certainly suitable for the highly nonlinear SDEs.

The rest of the paper is organized as follows. Section 2 gives some notation and preliminary results on the numerical solution of the truncated EM method. Section 3 begins to demonstrate the improved convergence rate in a finite time interval. Section 4 goes further to compare our result with the previous convergence rate results. Section 5 makes use of the truncated EM method to approximate the asymptotic stability. Section 6 concludes our main results. The Appendix proves that the classical EM method cannot reproduce asymptotic stability while the truncated method does.

2. Notation and lemmas

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ denote the probability expectation with respect to $\mathbb{P}$. Let $B(t)$ be an m-dimensional Brownian motion defined on the probability space and is $\mathcal{F}_t$-adapted. If A is a vector or matrix, its transpose is denoted by A^T. If $x \in \mathbb{R}^d$, then $|x|$ is the Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^TA)}$ be its trace norm. Moreover, for two real numbers a and b, we use $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. For a set G, its indicator function is denoted by I_G, namely $I_G(x) = 1$ if $x \in G$ and 0 otherwise.

Consider a d-dimensional nonlinear SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \quad t \geq 0, \tag{2.1}$$

with the initial value $x(0) = x_0 \in \mathbb{R}^d$, where $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Borel measurable. We impose two standing hypotheses in this paper.

Assumption 2.1. Assume that the coefficients f and g satisfy the local Lipschitz condition: For any $R > 0$, there is a $K_R > 0$ such that

$$|f(x) - f(y)| \vee |g(x) - g(y)| \leq K_R|x - y| \tag{2.2}$$

for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$.

Assumption 2.2. Assume that the coefficients satisfy the Khasminskii-type condition: There is a pair of constants $p > 2$ and $K > 0$ such that

$$x^Tf(x) + \frac{p-1}{2}|g(x)|^2 \leq K(1 + |x|^2) \tag{2.3}$$

for all $x \in \mathbb{R}^d$.

We state a known result (see, e.g., [18,19]) as a lemma for the use of this paper.

Lemma 2.3. *Under Assumptions 2.1 and 2.2, the SDE (2.1) has a unique global solution $x(t)$ and, moreover,*

$$\sup_{0\leq t\leq T} \mathbb{E}|x(t)|^p < \infty, \quad \forall T > 0. \tag{2.4}$$

Recall the truncated EM numerical scheme defined in [14]. We first choose a strictly increasing continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{|x|\leq u}\big(|f(x)| \vee |g(x)|\big) \leq \mu(u), \quad \forall u \geq 1. \tag{2.5}$$