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Circular sector area preserving approximation of circular arcs by geometrically smooth parametric polynomials

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ABSTRACT

The quality of the approximation of circular arcs by parametric polynomials is usually measured by the Hausdorff distance. It is sometimes important that a parametric polynomial approximant additionally preserves some particular geometric property. In this paper we study the circular sector area preserving parametric polynomial approximants of circular arcs. A general approach to this problem is considered and corresponding (nonlinear) equations are derived. For the approximants possessing the maximal order of geometric smoothness, a scalar nonlinear equation is analyzed in detail for the parabolic, the cubic and the quartic case. The existence of the admissible solution is confirmed. Moreover, the uniqueness of the solution with the optimal approximation order with respect to the radial distance is proved. Theoretical results are confirmed by numerical examples.

1. Introduction

The approximation of circular arcs is an important and well studied problem arising from practical applications as well as from theoretical problems. It is well known that circular arcs can be exactly represented by rational parametric curves but they do not possess the exact parametric polynomial representation. Thus the optimal approximation of circular arcs by parametric polynomials is a theoretical, as well as a practical issue, since some control systems still use parametric polynomials as the basic objects for the approximation and interpolation. Almost all research on this topic were concentrated in looking for good (or in some sense optimal) approximation schemes based on the Hausdorff distance or its simplifications, the radial distance, e.g. In this case the error is determined as the Euclidean distance between a point on the circular arc and a point on the parametric polynomial curve in the radial direction. It has been observed that the radial distance is (under some conditions) actually the Hausdorff distance (see [1] or [2], e.g.). This ensures that good approximants according to the radial distance are the optimal ones.

A more systematic approach to the approximation of circular arcs can be traced back for more than thirty years. In [3], the authors considered several G^1 and G^2 continuous approximations of circular arcs by cubic Bézier curves. The best uniform approximation by parabolic parametric polynomials was considered in [4] and similar problem for cubic curves was studied in [5]. Later on, a lot of results related to several types of geometric continuity interpolants of low degree appeared in the literature. In [1], the authors analyzed the optimal approximation by several types of quartic and quintic Bézier curves. A detailed analysis of geometric quintic approximants can be found in [6]. Some new cubic and quartic approximants were later studied in [7–12]. Approximation of the whole unit circle by parametric polynomials was considered in [13,14] and improved in [2].

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Authors in all these publications considered the Hausdorff distance and its simplifications as error measures. In [5] and in [6] the authors mentioned that also some other error measures could be used. They suggested to study the curvature profile of an approximant or the interpolation of the circular sector area (the area, defined by the circular arc and two boundary radial rays starting at the origin and passing through the boundary points of the arc). The first problem was partially considered in [15]. For the second issue it seems that there is no appropriate systematic study available and we shall consider it here. Thus the main purpose of this paper is to construct parametric polynomial approximants of circular arcs preserving the circular sector area. This might be of some practical importance for the numerical control of cutting machines if the preservation of the circular sector area is of great importance [16,17]. However, even in this case one does not want to lose high approximation order with respect to the Hausdorff distance. Ideally, the optimal approximation order should be preserved.

The paper is organized as follows. In the following section we present the interpolation problem in detail. First we consider the computation of the sector area defined by a parametric polynomial in the Bézier form. Then we study the conditions implying a geometric interpolant of maximal smoothness and provide the nonlinear scalar equation characterizing the solution. In Sections 3–5 we consider some particular cases of low degree and maximal geometric smoothness. A detailed analysis of the existence and the uniqueness of the optimal solution is provided for the parabolic G^0, the cubic G^1 and the quartic G^2 interpolants. Optimality of the solution is based on the radial distance function and numerical examples are presented. The paper is concluded with some future work plan in Section 6.

2. The interpolation problem

In this section a general problem of the interpolation of the circular sector area based on Bézier curves will be considered. Since a general circular arc can be translated, rotated and scaled to the unit circular arc centered at the origin and symmetric with respect to the first coordinate axis, and the same transformations can be used on a parametric polynomial approximant preserving interpolation and approximation properties, we can concentrate on symmetric unit circular arcs only. Thus, let $\boldsymbol{c} : [-\varphi, \varphi] \to \mathbb{R}^2$, $\varphi \in (0, \pi/2)$, be given unit circular arc centered at the origin. Consider its standard parameterization $\boldsymbol{c}(\alpha) = (\cos\alpha, \sin\alpha)^T$, $\alpha \in [-\varphi, \varphi]$. As we pointed out in the introduction, the standard approximation of circular arcs by parametric polynomials considers the radial distance as a measure of quality of the approximation. Here we consider an additional interesting measure, the interpolation of the circular sector area defined by the given circular arc $\boldsymbol{c}$. Our goal is to find a parametric polynomial circular arc approximation which exactly interpolates the given circular sector area. Moreover, we will require that the interpolating parametric polynomial will be the geometric approximant of the circular arc $\boldsymbol{c}$.

It is convenient to use the Bézier representation of a parametric polynomial, i.e.,

$$\boldsymbol{p}_n(t) = \sum_{j=0}^{n} \boldsymbol{b}_j B_j^n(t), \quad t \in [-1, 1], \tag{1}$$

where B_j^n, $j = 0, 1, \ldots, n$, are (reparameterized) Bernstein polynomials, given as

$$B_j^n(t) = \binom{n}{j} \left(\frac{1+t}{2}\right)^j \left(\frac{1-t}{2}\right)^{n-j}$$

and $\boldsymbol{b}_j \in \mathbb{R}^2$, $j = 0, 1, \ldots, n$, are control points. In order to incorporate some geometric properties of $\boldsymbol{p}_n$, we introduce the following definition.

Definition 1. A circular arc $\boldsymbol{c}$ and a parametric polynomial curve $\boldsymbol{p}_n$ have the geometric contact of order $k \in \mathbb{N}$ at the boundary points $\boldsymbol{c}(\pm\varphi)$, if there exists a smooth regular reparameterization $\rho : [-1, 1] \to [-\varphi, \varphi]$ with $\rho'(\pm 1) > 0$, such that

$$\frac{d^j \boldsymbol{p}_n}{dt^j}(\pm 1) = \frac{d^j(\boldsymbol{c} \circ \rho)}{dt^j}(\pm 1), \quad j = 0, 1, \ldots, k.$$

We say that $\boldsymbol{p}_n$ is the G^k approximation of $\boldsymbol{c}$ in this case.

It is well known that the G^0 approximation is equivalent to the interpolation of boundary points, the G^1 approximation additionally requires that $\boldsymbol{c}$ and $\boldsymbol{p}_n$ share the same unit tangents at the boundary points, etc.

The area of the circular sector, given by $\boldsymbol{c}$, is clearly φ, and the area of the sector, given by $\boldsymbol{p}_n$, is well known to be

$$a(\boldsymbol{p}_n) = \frac{1}{2} \int_{-1}^{1} \boldsymbol{p}_n(t) \times \boldsymbol{p}_n'(t) dt, \tag{2}$$

where $\boldsymbol{u} \times \boldsymbol{v} = u_1 v_2 - u_2 v_1$ is the standard planar cross product, defined for two planar vectors $\boldsymbol{u} = (u_1, v_1)^T$, $\boldsymbol{v} = (v_1, v_2)^T \in \mathbb{R}^2$. By some basic properties of (reparameterized) Bernstein polynomials and Bézier curves (see [18] and [19], e.g.) the relation (2) can be further simplified to

$$a(\boldsymbol{p}_n) = \frac{n}{4} \int_{-1}^{1} \left(\sum_{i=0}^{n} B_i^n(t) \boldsymbol{b}_i \right) \times \left(\sum_{j=0}^{n-1} B_j^{n-1}(t) \Delta \boldsymbol{b}_j \right) dt \tag{3}$$