Journal of Computational and Applied Mathematics 336 (2018) 235–248

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A combined finite element method for elliptic problems posted in domains with rough boundaries

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ARTICLE INFO

Article history:
Received 16 February 2017
Received in revised form 28 October 2017

MSC:
34E13
65N15
65N30

Keywords:
Combined finite element method
Rough boundary
Penalty technique

ABSTRACT

A combined finite element method is presented in this paper to solve the elliptic problems posted in domains with rough boundaries. Solving these problems numerically is difficult because resolving the boundaries usually requires very fine meshes, while good quality meshes often over-refine unnecessarily the interior of the domain. The basic idea of the proposed method is to use a fine mesh with size h in the vicinity of oscillating boundaries and a coarse mesh with size $H \gg h$ for other portions of the domain to reduce some unnecessary computational effort. The transmission conditions across the fine-coarse mesh interface are treated by the penalty technique. The key point of the method lies in the new scheme employing a weighted average in the definition of the bilinear form, which avoids the affection of the ratio H/h in the error estimate. We prove a quasi-optimal convergence in terms of elements since there is no whole H^2 regularity in the domain with rough boundaries. Numerical results are provided for elliptic equations in domains with non-oscillating or oscillating boundaries to illustrate the theoretical results.

1. Introduction

Many problems arising in modern material sciences and engineering are described by partial differential equations in domains with very rough boundaries. Typical examples include the electromagnetic scattering by an obstacle coated with an absorbing inhomogeneous paint, the dynamics of two-fluid flow in porous media and past rough walls, the hydrodynamic lubrication of rough surfaces, and among many others (see [1–3] and the references therein). Solving these problems numerically can be hard because resolving the boundaries usually requires very fine meshes and hence tremendous amount of computer memory and CPU time. To overcome this difficulty, many papers have been devoted to the homogenization of boundary value problems in domains with fast oscillating boundaries (see [4–6,1] and the references therein). In general, the homogenized problems are the boundary value problems for the same equations in the same domains but with the mollified boundary instead of the oscillating one. The mollified boundary and the effective boundary condition on it are determined by the original boundary condition and the geometry of the oscillations.

The homogenization theory mainly studied the identification of the homogenized problems and proving the convergence theorems for the solutions. To our knowledge, less work of multiscale methods has been done for the boundary value problems in a domain with multiscale boundary. Only recently, some researchers began to concern about this topic. See, for example, the multiscale finite element method (MsFEM) for Laplace equation with homogeneous Dirichlet boundary value on rough domain [7], the MsFEM for Laplace equation with oscillating Neumann boundary conditions on rough boundaries [8], and the multiscale methods based on the localized orthogonal decomposition (LOD) technique for problems

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https://doi.org/10.1016/j.cam.2017.12.049

with complex geometry [9]. Just as the classical MsFEM [10], the author in [7] defines the multiscale basis functions for the elements near the rough boundary by solving a cell problem with the homogeneous Dirichlet condition on the rough edge and with linear nodal basis function as boundary condition on other edges. By this way, the influence of the geometry should be captured by the basis functions. The MsFEM in [8] introduces a special Neumann boundary condition that incorporates both the microscopically geometrical and physical information of the rough boundary for the local cell problem posed on elements with rough edge. Compared to the method in [7], this approach can be applied to problems with non-Dirichlet boundary conditions or problems with inhomogeneous Dirichlet boundary value over the rough boundary. However, the analysis in [7,8] is limited to periodic data. In [9], the multiscale analysis based on the LOD technique is extended to elliptic problems on domains with cracks or complicated boundary, in which corrected coarse test and trail spaces taking the fine features of the domain into account are constructed. We remark that all of the above mentioned methods are based on solutions of local problems which can be computed off-line and can thus be done in parallel. However, these methods are still expensive since local problems depend on the small parameter that characterizes the roughness of the boundary. Of course, all these methods are much cheaper than using pure piecewise linear FEM in the whole domain.

The oscillation of the boundary also stands for a kind of scale, which are subject to multiscale problems (see [11,9,8]). For such problem, since the global regularity of the solution is low (especially near the boundary), the traditional finite element method (FEM) becomes inapplicable because resolving the boundaries usually requires very fine meshes, and good quality meshes often over-refine unnecessarily the interior of the domain. Thus, in this paper, we try to introduce a combined finite element method (combined FEM) which uses a fine mesh with size h in the vicinity of oscillating boundary while uses a coarse mesh with size $H(\gg h)$ for the interior subdomains. By this way, we can cut down some unnecessary computational effort. The negative effect of the processing method is that the generated mesh has many hanging nodes along the fine-coarse mesh interface. For instance, here for each edge of the interface coarse-element, it has $(H/h-2)$ hanging nodes. Along the interface, the numerical solution is discontinuous. Thanks to the penalty techniques used in the interior penalty discontinuous (or continuous) Galerkin methods originated in 1970s [12–17], we may deal with the transmission condition across the fine-coarse mesh interface by penalizing the jumps from the function values of the finite element solution on the fine mesh to those on the coarse mesh. However, if we use the traditional variational form with the arithmetic average of the function values from the fine and coarse grids respectively, the error analysis shows that the ratio H/h debases the convergence rate (see [16,18]). Hence, in the proposed scheme, we employ a weighted average in the definition of the bilinear form, which can eliminate the affection of the ratio H/h in the error estimates. The weighted coefficients are depended on the sizes of coarse and fine meshes along the interface, namely H and h. The penalty coefficient is defined as $\gamma/(H+h)$ for some positive constant γ. We prove a quasi-optimal convergence in terms of elements since for the rough boundary problem its solution is generally in $H^s(\Omega)$ with $1<s<2$.

There are some other numerical methods to handle the complex geometrical boundary such as the cut finite element method in [19] where the boundary and interface conditions are built into the discrete formulation by the Nitsche's method, the extended/generalized finite element method in [20] which is achieved by adding special shape functions to the polynomial approximation space of the classical finite element method, and the composite finite elements in [21]. The composite finite element method which constructs a coarse basis that is fitted to the boundary has been successfully applied to problems in domain with oscillating boundaries (see [22–24]).

Besides problems with oscillating boundaries, our method may be applied into other partial differential equations with challenging singularities, e.g., well problems with the Dirac function singularities, partial differential equations on domains containing small geometric details, high conductivity channels which appear in many fields of science and engineering. It is also possible to combine the present method with the standard MsFEM to deal with the problems with both oscillating coefficients and oscillating boundary data. Note that a lot of ways have been developed to deal with the multiscale problems with singularities, such as, the MsFEMs (see [25–31]) and the localized orthogonal decomposition methods (see [32–35]). The proposed method is very similar to the combined MsFEM introduced in [30], which uses the traditional linear FEM directly on a fine mesh of the problematic part of the domain and the oversampling MsFEM on a coarse mesh of the other part to solve the multiscale problems with singularities.

The outline of the paper is as follows. In Section 2, we describe the model problem and introduce the combined FEM. In Section 3, we analyze the proposed method, including the continuity, the coercivity of the scheme, and the bound of the error in energy norm which shows that there is no ratio H/h appearing in the error estimate and the convergence order is quasi-optimal with respect to elements. In Section 4, we simulate some model problems in the domain with rough boundaries by the combined FEM. The numerical experiments verify the theoretical results. We end with some conclusions which are drawn in Section 5.

Throughout the paper, C, γ' and γ are used to denote the generic positive constants which are independent of H, h and maybe depend on some constant parameters (e.g., penalty parameter γ), which are different in different places. We also use the shorthand notation $A \lesssim B$ and $B \gtrsim A$ for the inequality $A \le CB$ and $B \ge CA$ respectively. The notation $A \eqsim B$ is equivalent to the statement $A \lesssim B$ and $B \lesssim A$.