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An analytical approach: Explicit inverses of periodic tridiagonal matrices

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Abstract

We derive an explicit formula for the inverse of a general, periodic, tridiagonal matrix. Our approach is to derive its *LU* factorization using backward continued fractions (BCF) which are an essential tool in number theory. We then use these formulae to construct an algorithm for inverting a general, periodic, tridiagonal matrix which we implement in Maple[1]. Finally, we present the results of testing the efficiency of our new algorithm against another published implementation and against the library procedures available within Maple to invert a general matrix and to compute its determinant.

Keywords: Matrix inversion, LU-Factorization, Inverse, Backward continued fraction

1. Introduction

Tridiagonal matrices have been frequently used in various application areas ranging from engineering to economics (see [1, 2, 3, 4, 5, 6, 7, 8]) as well as in the computation of special functions, PDEs and number theory (see [9, 10, 11, 12, 13, 14]). Various features of tridiagonal matrices are used to solve the systems of linear equations that arise from these applications (see, for example, [10, 12]) and many authors (for example, [15, 16, 17, 18, 5, 19, 20, 21, 22]) have studied various tridiagonal matrices and their properties such as LU decompositions, determinants and inverses.

Define the $n \times n$ periodic tridiagonal matrix $G(\delta, \mu)$, or briefly G, with $G_{kk} = x_k$ for $1 \leq k \leq n$, $G_{k+1,k} = z_k$ and $G_{k,k+1} = y_k$ for $1 \leq k \leq n-1$, $G_{1n} = \delta$, $G_{n1} = \mu$ and 0 otherwise. This has the form

$$G = \begin{bmatrix} x_1 & y_1 & & & \delta \\ z_1 & x_2 & y_2 & & \\ & z_2 & x_3 & \ddots & \\ & & \ddots & \ddots & y_{n-1} \\ \mu & & & z_{n-1} & x_n \end{bmatrix}. \tag{1.1}$$

Many methods have been reported for solving systems of linear equations whose coefficient matrix is periodic tridiagonal; see, for example, [23, 24, 25, 26, 27]. The efficacy of a particular algorithm is often dependent both on the type of data used to define G (floating point numbers, rational values or algebraic expressions) and the required result (inverse matrix, solution of a linear system, determinant, etc.).

For some particular forms of (1.1) simple closed form inverses are known; see, for example, [28, 17, 29, 13, 22]. However, in general, such closed forms are not available and obtaining exact inverses to (1.1) efficiently, when the elements are rational numbers or algebraic quantities, requires a symbolic algebra (SA) system like Maple [30] or Mathematica [31]. Experience with computing the inverse of (1.1) when the elements are floating point values suggests that we should be able to construct SA algorithms which are far more efficient that a general matrix inversion procedure by taking account of the special structure.

When $\delta = \mu = 0$, the periodic, tridiagonal matrix G is reduced to the general tridiagonal matrix T. Mallik [29] gives an explicit formula based on linear difference equations for the elements of T^{-1}.

[1] Maple is a trademark of Waterloo Maple Inc.

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