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A closed-form pricing formula for European options under the Heston model with stochastic interest rate

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ABSTRACT

In this paper, a closed-form pricing formula for European options in the form of an infinite series is derived under the Heston model with the interest rate being another random variable following the CIR (Cox–Ingersoll–Ross) model. One of the main advantages for the newly derived series solution is that we can provide a radius of convergence, which is complemented by some numerical experiments demonstrating its speed of convergence. To further verify our formula, option prices calculated through our formula are also compared with those obtained from Monte Carlo simulations. Finally, a set of pricing formulae are derived with the series expanded at different points so that the entire time horizon can be covered by converged solutions.

1. Introduction

In 1973, Black & Scholes [1] made a breakthrough by proposing an elegant model with the underlying price following a geometric Brownian motion and deriving an analytical formula for European option prices. However, some of its simplified assumptions made to achieve the analytical simplicity and tractability are inappropriate and can cause mis-pricing problems. In particular, one of its main drawbacks is the constant volatility assumption, which contradicts to the phenomenon of "volatility smile" [2] observed in real markets that the implied volatility extracted from real market data tends to exhibit a "smile" curve. As a result, a number of modifications have been proposed to incorporate the non-constant volatility into the Black–Scholes model.

Non-constant volatility models can mainly be divided into two categories, i.e., local volatility and stochastic volatility. The former, assuming that the volatility be a deterministic function of the underlying price and time, is considered by Dupire [3], Derman & Kani [4] and Rubinstein [5]. Unfortunately, empirical studies have already suggested that the "smile dynamics" are poorly captured by local volatility models (e.g., Hagan et al. [6]). Therefore, the latter category, making the volatility of the underlying price another random variable, has thus become much more popular.

However, due to the addition of another stochastic source, it is very difficult to derive analytical solutions for most of stochastic volatility models, and numerical methods must be resorted to in these cases. For example, Johnson [7] and Scott [8] directly simulated the stochastic processes with the Monte Carlo simulation technique, while Wiggins [9] adopted the finite difference method to solve the PDEs (partial differential equations) governing option prices. Unfortunately, a noticeable pity for numerical methods is always the lack of speed in computation, which makes it difficult to implement these models in real markets since model calibration is very time-consuming and the lack of analytical pricing formula can make the situation even worse. Therefore, further research interest was led to finding more appropriate stochastic volatility processes with analytic pricing formula for European call options. Specifically, Hull & White [10] proposed that the volatility follow

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another geometric Brownian motion and derived a power series solution for option prices. Albeit appealing, the assumption of the zero correlation between the underlying price and the volatility made in their model is at odds with the so-called "leverage effects" that the underlying price and the volatility should be negatively correlated [11]. Moreover, although an Ornstein–Uhlenbeck process is adopted for the volatility process in the Stein–Stein model [12] and a closed-form pricing formula is presented, some of the model flaws, such as unable to prevent the volatility from going negative, make this model still unsatisfactory. In 1993, Heston [13] contributed a lot to the literature by incorporating the CIR (Cox–Ingersoll–Ross) model to describe the volatility process and deriving a closed-form pricing formula for European options. Two main reasons can account for its popularity; one is that the volatility process itself satisfies a wide range of basic properties, such as the obvious non-negative property and the mean-reverting property being consistent with the results of empirical studies [14], and another is that there exists a closed-form formula when pricing options, which can save us a lot of time and effort when conducting model calibration. Given the fact that the introduction of the stochastic volatility makes the markets incomplete and there exist different equivalent martingale measures, the option price under the Heston model is not unique, and the analytical pricing formula derived by Heston can no longer be used if a different martingale measure is chosen. Recently, He & Zhu [15,16] presented a different analytical pricing formula for European options under the Heston model by choosing the so-called minimal entropy martingale measure.

However, it should also be pointed out that the well-known Heston model is not perfect either in many senses (there may not even be a perfect one!) and many attempts are made to improve its pricing performance in real markets, such as the introduction of the time-dependent Heston models [17] and the regime-switching Heston models [18]. One of the most popular approaches is to incorporate the stochastic interest rate into stochastic volatility models to form a hybrid model since there is a lot of empirical evidence suggesting that introducing stochastic interest rate into option pricing models can lead to better model performance [19,20], and a number of authors have worked on this area. For instance, a combination of the correlated Stein–Stein model [21] and the Hull–White interest rate model [22] is adopted in [23] with European options evaluated under the Fourier cosine expansion framework. Furthermore, approximation formulae for European option prices are presented when the underlying price follows the Heston stochastic volatility model with the interest rate described by the CIR model [24].

In this paper, we adopt the Heston-CIR hybrid model for the underlying price and we aim to present a closed-form pricing formula for European options as models with exact and analytical solutions are much more favoured in real markets. Based on the technique of numeraire change, we firstly obtain a general pricing formula with the unknown characteristic function of the underlying price under a forward measure. Then, the target characteristic function is analytically worked out written in the form of an infinite series by expending the solution in terms of the time to expiry; such a series solution is accompanied by a proof of convergence that a radius of convergence is theoretically figured out. Numerical experiments are carried out to show the convergence speed as well as the accuracy of the newly derived formula. Finally, for the situation that the time to expiry is larger than the provided radius of convergence, we have also come out an alternative way by deriving a set of pricing formulae converging on a particular region with different expansion points so that the entire time horizon can be covered by converged solutions.

The rest of the paper is organized as follows. In Section 2, a brief introduction of the Heston-CIR hybrid model is provided. In Section 3, we firstly introduce a general pricing approach, after which we present an analytical pricing formula in a series form based on the change of numeraire and the derivation of the characteristic function. A radius of convergence for this series solution is subsequently provided, and numerical experiments and examples are also presented. In Section 4, a note on how to deal with the situation when the time to expiry exceeds the provided radius of convergence is presented, followed by some concluding remarks given in Section 5.

2. The Heston-CIR hybrid model

In this section, we will mainly discuss the specific model we adopt for European option pricing. Although the Black–Scholes model is very popular among market traders, some of the unrealistic assumptions made to achieve analytical tractability are inappropriate, such as the constant volatility assumption [2] and the constant interest rate assumption [19]. As a result, a number of modifications to the Black–Scholes model have been proposed to incorporate the effect of stochastic volatility and stochastic interest rate [12,25]. One of the most popular models belonging to the category of stochastic volatility is the so-called Heston model, and there are mainly two reasons that can account for this. One is that the dynamics of the Heston model satisfy several properties, such as the non-negative property and the mean-reverting property being consistent with real market observations, and another is that there exists a closed-form pricing formula for European options, which makes it easy to be implemented in real markets. What we adopt here is actually a hybrid model combining the Heston stochastic volatility model and the CIR stochastic interest rate model, the dynamics of which under a risk-neutral measure $\mathbb{Q}$ are specified as

$$\begin{bmatrix} \frac{dS_t}{S_t} \\ dv_t \\ dr_t \end{bmatrix} = \mu^{Q} dt + \Sigma \times C \times \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ dW_{3,t} \end{bmatrix}, \tag{2.1}$$