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Affine matrix rank minimization problem via non-convex fraction function penalty

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ABSTRACT

Affine matrix rank minimization problem is a fundamental problem in many important applications. It is well known that this problem is combinatorial and NP-hard in general. In this paper, a continuous promoting low rank non-convex fraction function is studied to replace the rank function in this NP-hard problem. An iterative singular value thresholding algorithm is proposed to solve the regularization transformed affine matrix rank minimization problem. With the change of the parameter in non-convex fraction function, we could get some much better results, which is one of the advantages for the iterative singular value thresholding algorithm compared with some state-of-art methods. Some convergence results are established. Moreover, we proved that the value of the regularization parameter $\lambda > 0$ cannot be chosen too large. Indeed, there exists $\bar{\lambda} > 0$ such that the optimal solution of the regularization transformed affine matrix rank minimization problem is equal to zero for any $\lambda > \bar{\lambda}$. Numerical experiments on matrix completion problems and image inpainting problems show that our method performs effective in finding a low-rank matrix compared with some state-of-art methods.

1. Introduction

In recent years, affine matrix rank minimization (AMRM) problem has attracted much attention in many important applications such as collaborative filtering in recommender systems [1,2], minimum order system and low-dimensional Euclidean embedding in control theory [3,4], network localization [5], and so on. It can be viewed as the following mathematical form

$$(\mathrm{AMRM}) \qquad \min_{X\in\Re^{m\times n}} \ \mathrm{rank}(X) \ \ s.t. \ \ \mathcal{A}(X)=b \tag{1}$$

where the linear map $\mathcal{A}:\Re^{m\times n}\mapsto\Re^{d}$ and the vector $b\in\Re^{d}$ are given. Without loss of generality, we assume $m\leq n$. The matrix completion problem

$$\min_{X\in\Re^{m\times n}} \ \mathrm{rank}(X) \ \ s.t. \ \ X_{i,j}=M_{i,j}, \ \ (i,j)\in\Omega \tag{2}$$

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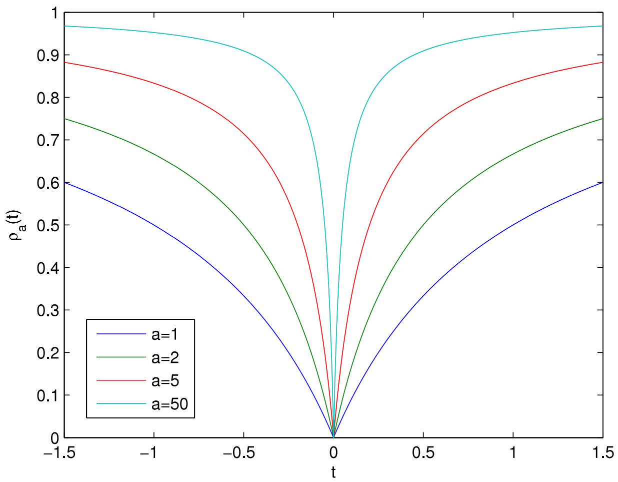

Fig. 1. The behavior of the fraction function $\rho_a(t)$ for various values of a.

is a special case of the problem (AMRM), where X and M are both $m \times n$ real matrices, the set Ω is a subset of indexes set of all pairs (i, j), and the subset $\{M_{i,j}|(i, j) \in \Omega\}$ of the entries is known. If the projector $\mathcal{P}_\Omega : \Re^{m\times n} \to \Re^{m\times n}$ is defined as

$$[\mathcal{P}_\Omega X]_{i,j} = \begin{cases} X_{i,j}, & \text{if } (i, j) \in \Omega; \\ 0, & \text{if } (i, j) \notin \Omega, \end{cases} \tag{3}$$

and the resulting matrix is $M_\Omega = \mathcal{P}_\Omega X$, then the matrix completion problem can be reformulated as

$$\min_{X\in\Re^{m\times n}} \operatorname{rank}(X) \ \ s.t. \ \mathcal{P}_\Omega X = M_\Omega. \tag{4}$$

In general, however, the problem (AMRM) is a challenging nonconvex optimization problem and is known as NP-hard [6]. Nuclear-norm affine matrix rank minimization problem (NuAMRM) is the most popular alternative [1,4,6–9], and the minimization has the following form

$$\text{(NuAMRM)} \qquad \min_{X\in\Re^{m\times n}} \|X\|_* \ \ s.t. \ \mathcal{A}(X) = b \tag{5}$$

for the constrained problem and

$$\text{(RNuAMRM)} \qquad \min_{X\in\Re^{m\times n}} \left\{ \|\mathcal{A}(X) - b\|_2^2 + \lambda\|X\|_* \right\} \tag{6}$$

for the regularization problem, where $\lambda > 0$ is the regularization parameter, $\|X\|_* = \sum_{i=1}^{m} \sigma_i(X)$ is the nuclear-norm of matrix X, and $\sigma_i(X)$ presents the ith largest singular value of matrix X arranged in descending order.

As the compact convex relaxation of the NP-hard problem (AMRM), the problem (NuAMRM) possesses many theoretical and algorithmic advantages [10–13]. However, it may be suboptimal for recovering a real low-rank matrix. In fact, the problem (NuAMRM) may yield a matrix with much higher rank and need more observations to recover a real low-rank matrix [1,11]. Moreover, the problem (RNuAMRM) tends to lead to biased estimation by shrinking all the singular values toward zero simultaneously, and sometimes results in over-penalization as the l_1-norm in compressed sensing [14]. With recent development of non-convex relaxation approach in sparse signal recovery problems, many researchers have shown that using a continuous non-convex function to approximate the l_0-norm is a better choice than using the l_1-norm (see, e.g., [15–26]). Meanwhile, some empirical evidence (see, e.g., [27–32]), has shown that, the non-convex algorithms can really make a better recovery in some matrix rank minimization problems. This brings our attention back to the non-convex optimizations.

Inspired by the good performance of the non-convex fraction function in image restoration (see [33]), in this paper, we replace the rank function rank(X) in problem (AMRM) with a continuous promoting low rank non-convex function:

$$P_a(X) = \sum_{i=1}^{m} \rho_a(\sigma_i(X)) \tag{7}$$

where the non-convex function

$$\rho_a(t) = \frac{a|t|}{a|t| + 1} \tag{8}$$