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Adaptive timestepping for pathwise stability and positivity of strongly discretised nonlinear stochastic differential equations

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ABSTRACT

We consider the use of adaptive timestepping to allow a strong explicit Euler–Maruyama discretisation to reproduce dynamical properties of a class of nonlinear stochastic differential equations with a unique equilibrium solution and non-negative, non-globally Lipschitz coefficients. Solutions of such equations may display a tendency towards explosive growth, countered by a sufficiently intense and nonlinear diffusion.

We construct an adaptive timestepping strategy which closely reproduces the almost sure (a.s.) asymptotic stability and instability of the equilibrium, and which can ensure the positivity of solutions with arbitrarily high probability. Our analysis adapts the derivation of a discrete form of the Itô formula from Appleby et al. (2009) in order to deal with the lack of independence of the Wiener increments introduced by the adaptivity of the mesh. We also use results on the convergence of certain martingales and semi-martingales which influence the construction of our adaptive timestepping scheme in a way proposed by Liu & Mao (2017).

1. Introduction

Consider the scalar stochastic differential equation (SDE) of Itô type

$$dX(t) = X(t)f(X(t))dt + X(t)g(X(t))dW(t), \quad t \geq 0, \qquad (1)$$
$$X(0) = \varsigma \geq 0,$$

where $(W(t))_{t\geq 0}$ is a one-dimensional Wiener process; let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of W. The drift and diffusion coefficients satisfy:

Assumption 1. Let $f, g : \mathbb{R} \to [0, \infty)$ be non-negative functions such that $g(u) \neq 0$ for $u \neq 0$.

In this article, we use an adaptive timestepping strategy to reproduce qualitative properties of solutions of (1) in an explicit strong Euler–Maruyama discretisation given by

$$X_{n+1} = X_n \left(1 + h_n f(X_n) + g(X_n)\left[W(t_n) - W(t_{n-1})\right]\right), \quad n \in \mathbb{N}, \qquad (2)$$

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with $X_0 = X(0) = \varsigma$, $t_{-1} = 0$. Each h_n is one of a sequence of random timesteps, generated as a function of X_n, and we set $\{t_n := \sum_{i=1}^{n} h_i\}_{n\in\mathbb{N}}$.

Our first goal is to design a strategy that allows discretisations of the form (2) to closely reproduce the almost sure (a.s.) stability and instability of the unique equilibrium solution $X(t) \equiv 0$ of (1). The strategy will be required to capture the stabilising effect of the diffusion as it counters the tendency towards explosive growth due to the positive drift. Since we will use martingale and semimartingale convergence results in our analysis, we must adopt elements of the approach developed by Liu & Mao [1] in order to ensure that those results are applicable.

Our second goal is to investigate the effect of our adaptive timestepping strategy on the probability of positivity of solutions of (2). Unique solutions of (1) when $\varsigma > 0$ are necessarily positive, though a highly nonlinear diffusion coefficient makes it likely that trajectories of a fixed-step discretisation will overshoot the equilibrium and become negative. Adaptive timestepping was successfully used in [2] to preserve positivity with high probability in equations with either a dominant nonlinear and strongly zero-reverting drift coefficient, or a dominant and highly variable nonlinear diffusion coefficient. That article was a follow up to [3], and our analytic technique is adapted from both.

An analysis of the ability of explicit numerical methods with adaptive timesteps to reproduce the dynamics of solutions of (1) is important because explicit Euler methods of the form (2) with constant stepsize $h_n \equiv h$ are known (see [4]) to fail to converge strongly to solutions of (1) if either f or g grows superlinearly, as is the case for (5). Fixed-step taming methods were introduced first in [5] to provide an alternative class of strongly convergent explicit methods for such equations, but may not provide an optimal reproduction of qualitative behaviour: see [6,7]. The semi-discrete method proposed in [8] and applied to equations with super-linear coefficients in [9] succeeds in preserving positivity. Drift implicit methods in combination with an appropriate transform have also been shown in [10] to preserve the domain of solutions of SDEs with both sublinear and superlinear coefficients.

It was recently shown (see [7,11]) that, for equations with one-sided Lipschitz drift and globally Lipschitz diffusion coefficients, adaptive timestepping strategies can be used to ensure strong convergence of solutions of the explicit Euler method with variable stepsizes, and therefore their effect on the dynamics of solutions is of interest: see [12].

Let us now consider a minimal set of additional constraints to place upon f and g. Suppose first that f and g are locally Lipschitz continuous and that Assumption 1 holds. Then there exists a unique, continuous $\mathcal{F}_t$-measurable process X (see [13,14]) satisfying (1) on the interval $[0, \tau_e^{\varsigma})$, where $\tau_e^{\varsigma} = \inf\{t > 0 : |X(t,\varsigma)| \notin [0,\infty)\}$. Define the first hitting time of zero to be $\upsilon_e^{\varsigma} = \inf\{t > 0 : |X(t,\varsigma)| = 0.\}$. It was proved in [15] that $\upsilon_e^{\varsigma} = \tau_e^{\varsigma} = \infty$, and therefore unique positive solutions exist on all of $\mathbb{R}^+$, if

$$\sup_{u\neq 0} \frac{2f(u)}{g^2(u)} = \beta < 1. \tag{3}$$

Condition (3) is close to being sharp. (1) has an equilibrium solution $X(t) \equiv 0$, and (see [15]) if

$$\lim_{u\to 0} \frac{2f(u)}{g^2(u)} > 1,$$

then this equilibrium is a.s. unstable: for all $\varsigma > 0$,

$$\mathbb{P}\left[\lim_{t\to\infty} X(t) = 0\right] = 0.$$

Alternatively, if

$$\lim_{u\to 0} \frac{2f(u)}{g^2(u)} < 1, \tag{4}$$

then for all $\varsigma > 0$

$$\mathbb{P}\left[\lim_{t\to\infty} X(t) = 0\right] > 0.$$

Conditions (3) and (4) require the diffusion coefficient g to have a stabilising effect. For example, consider the scalar stochastic differential equation with positive polynomial coefficients

$$dX(t) = X(t)\left(X^{\nu}(t)dt + \sigma X^{\nu/2}(t)dW(t)\right), \quad t \geq 0, \quad X(0) = \varsigma \geq 0, \tag{5}$$

where $\nu \in (0,\infty)$. In this case $f(u) = u^{\nu}$ and $g(u) = \sigma u^{\nu/2}$, and therefore (3) is satisfied with $\lim_{u\to 0}\frac{2f(u)}{g^2(u)} = 2/\sigma^2 < 1$ when $\sigma^2 > 2$. So if the intensity of the stochastic perturbation is sufficiently large, unique positive solutions exist on $[0,\infty)$ and converge to zero with positive probability. If $\sigma = 0$, then (5) becomes the ordinary differential equation

$$x'(t) = [x(t)]^{1+\nu}, \quad t \geq 0, \quad x(0) = \varsigma > 0,$$